On the ferromagnetism equations in the non static case

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Contents

1 Introduction

In this paper we are interested in an initial boundary value problem for a mathematical model in ferromagnetism. The physical context is the following. A piece of ferromagnet is supposed to be a regular bounded open set Ω in $\mathbb{R}^3$. The magnetic state at a point $x \in \Omega$ at time t is given by a vector $u(t,x) \in \mathbb{R}^3$ which belongs to the unit sphere of $\mathbb{R}^3$, called the *magnetic moment*. The evolution of u is coupled to the evolution of the electromagnetic field $\big(E(t,x), H(t,x)\big)$ in the whole space $\mathbb{R}^3$, by a system of nonlinear partial differential equations.
The first equation is the following Landau-Lifschitz equation in $\mathbb{R}^+_t \times \Omega_x$, where ε^2 is supposed to be a constant :

$$\begin{cases} \partial_t u = u \wedge (H + \varepsilon^2 \Delta u) - u \wedge \big(u \wedge (H + \varepsilon^2 \Delta u)\big) \quad \text{in } [0,+\infty[\times\Omega \\ \partial_{\mathbf{n}} u = 0 \ \text{ in } \ [0,+\infty[\times\partial\Omega \\ u_{|t=0} = u_0 \ . \end{cases} \tag{1.1}$$

where $\mathbf{n}$ is the unitary outward normal at the boundary $\partial\Omega$.

This equation is coupled with the Maxwell system in $R^+_t \times \mathbb{R}^3$

$$\begin{cases} \partial_t (H + \bar{u}) + \text{curl } E = 0 \\ \partial_t E - \text{curl } H = 0 \\ (E, H)_{|t=0} = (E_0, H_0). \end{cases} \tag{1.2}$$

where $\bar{u}$ means the extension of u by 0 outside of $\mathbb{R} \times \Omega$.

Remark 1.1 *In all the paper we take all the physical constants equal to 1, excepted the exchange coefficient, since their value don't change the mathematical analysis of the equations.*

Furthermore, the solution must satisfy the divergence condition

$$\text{div } (H + \bar{u}) = 0, \tag{1.3}$$

and the constraint

$$|u(t,x)| = 1, \quad x \in \Omega, \ t \geq 0. \tag{1.4}$$

A basic observation is that these two last conditions are propagated by the full system, from the initial conditions. The condition (1.3) is given by the Maxwell equations (1.2), since the first equation of (1.2) implies that $\partial_t \text{div } (H + \bar{u}) = 0$. The condition (1.3) is then satisfied for all $t \geq 0$ if and only if it is satisfied for $t = 0$. In other words, condition (1.3) means exactly that the initial data H_0 and u_0 satisfy

$$\text{div } (H_0 + \overline{u_0}) = 0. \tag{1.5}$$

The same remark is true for the condition (1.4), assuming however that u is regular enough, since the equation (1.1) implies $\partial_t(|u(t,x)|^2) = 0$.

The existence of *global weak solutions* for the system (1.1), (1.2) was established by A. Visintin in [32], and for another form of the system (equivalent for regular enough solutions) by G. Carbou and P. Fabrie in [8].

In this paper, we first prove the existence and uniqueness of regular enough solutions for the system (1.1), (1.2). The solutions obtained are local in time. This result is stated in section 2. The section 3 is concerned with the question of the asymptotic behavior of the solution of (1.1)-(1.2) as $\varepsilon > 0$ tends to 0. ¿From a formal point of view, the system obtained when $\varepsilon = 0$, is equivalent to a first order semilinear symmetric hyperbolic system, which is known to admit local piecewise regular solutions (Sobolev regularity) discontinuous across the boundary $\mathbb{R} \times \partial\Omega$, which is a characteristic hypersurface of constant multiplicity for this hyperbolic system. This hyperbolic system has a very particular structure and admits global solutions as proved by Joly, Métivier and Rauch in [19]. We prove here two new results. First, a solution (u^0, E^0, H^0) of the limit hyperbolic system being given on $[0, T]$, we show that under some natural assumptions, this solution is limit of a family of solutions $(u^\varepsilon, E^\varepsilon, H^\varepsilon)$ of (1.1)-(1.2). The other result is that if u^0 satisfy the additional condition $\partial_{\mathbf{n}} u^0(0, .)_{|\partial\Omega} = 0$, the solution of (1.1)-(1.2) with initial data $(u^0{}_{|t=0}, E^0{}_{|t=0}, H^0{}_{|t=0})$ converges to (u^0, E^0, H^0) as ε goes to 0. To obtain this results, we perform an asymptotic expansion in ε and bring to the fore a boundary layer of characteristic size ε, and amplitude ε, localized closed to $\partial\Omega$. As it is classical in BKW method, we have to suppose that the limit solution (u^0, E^0, H^0) is very regular on each side of Ω (Sobolev piecewise regularity).

Notation. In all the paper, we will note $\mathbf{H}^m := (H^m)^3 = (W^{m,2})^3$ the usual Sobolev spaces of functions with values in $\mathbb{R}^3$, and $\mathbf{L}^p := (L^p)^3$ the usual Lebesgue spaces with values in $\mathbb{R}^3$.

2 A local existence result for a fixed $\varepsilon > 0$

Let us introduce some notations. For $T > 0$, let us call $\mathcal{A}(T)$ the set of functions

$$u \in L^2\big([0,T]; \mathbf{H}^3(\Omega)\big) \cap \mathcal{C}\big([0,T]; \mathbf{H}^2(\Omega)\big) \cap \mathcal{C}^1\big([0,T]; \mathbf{H}^1(\Omega)\big)$$

such that $\partial_t u \in L^2\big([0,T]; \mathbf{H}^2(\Omega)\big)$ and $\partial_t^2 u \in L^2([0,T] \times \Omega)$.
Concerning the regularity of the electromagnetic field, we will use the following classical space

$$\mathbf{H}_{\text{curl}} := \{v \in L^2(\mathbb{R}^3; \mathbb{R}^3) \text{ such that curl } v \in L^2(\mathbb{R}^3; \mathbb{R}^3)\}$$

equiped with the natural norm $\|v\|_{L^2} + \|\text{curl } v\|_{L^2}$. The main result of the section is the following.

Theorem 2.1 *Let $\varepsilon > 0$ be fixed. Let $u_0 \in \mathbf{H}^3(\Omega)$ satisfying $|u_0| = 1$, $\partial_{\mathbf{n}} u_{0|\partial\Omega} = 0$. Let $(E_0, H_0) \in \mathbf{H}_{\text{curl}} \times \mathbf{H}_{\text{curl}}$. Assume that* div $(H_0 + \bar{u_0}) = 0$. *Then there exists $T > 0$ and a unique solution (u, E, H) to the problem* (1.1), (1.2), *such that $u \in \mathcal{A}(T)$, and*

$$E, H \in \mathcal{C}^1\big([0,T]; \mathbf{L}^2(\mathbb{R}^3)\big) \cap \mathcal{C}\big([0,T] : \mathbf{H}_{\text{curl}}\big).$$

Furthermore, $|u| = 1$ in $[0,T] \times \Omega$ and div $(H + \bar{u}) = 0$ *for all $t \in [0,T]$.*

This theorem will be deduced from theorem 4.1 below, which is proved in section 5.

3 Asymptotic analysis as $\varepsilon \to 0$

In this section, we are interested in the behavior of the local solution described in theorem 2.1, as ε tends to 0. This is a natural question of current interest in the modelisation of micromagnetism.

Let us consider the system formally obtained when $\varepsilon = 0$, on a time interval $]0,T[$, which writes

$$\begin{cases} \partial_t u^0 = u^0 \wedge H^0 - u^0 \wedge (u^0 \wedge H^0) \text{ in }]0,T[\times\Omega \\ \\ \partial_t (H^0 + \overline{u^0}) + \text{curl } E = 0 \text{ in }]0,T[\times\mathbb{R}^3 \\ \\ \partial_t E^0 - \text{curl } H^0 = 0 \text{ in }]0,T[\times\mathbb{R}^3 \end{cases} \tag{3.1}$$

Note that the first equation holds in $]0,T[\times\Omega$, and that no boundary condition is needed on $]0,T[\times\partial\Omega$ for u^0. This system satisfies as the original (1.1)-(1.2) system, the propagation properties of $|u^0(t,x)|$ and div $(H^0 + \overline{u^0})$ in the sens that the relations

$$\begin{aligned} &|u^0(t,x)|^2 = 1, \forall x \in \Omega, \forall t \in [0,T] \\ &\text{div } (H^0 + \overline{u^0}) = 0, \forall t \in [0,T] \end{aligned} \tag{3.2}$$

hold if and only if they are satisfied at $t = 0$.
Now, since the principal part of the first equation is the field ∂_t, it follows that $(u^0, H^0, E^0) \in L^\infty_{loc}$ satisfies system (3.1) in the sens of distributions if and only if $(V^0 := \overline{u^0}, H^0, E^0)$ satisfies the following semilinear first order symmetric hyperbolic system in the domain $]0,T[\times\mathbb{R}^3$:

$$\begin{cases} \partial_t V^0 = V^0 \wedge H^0 - V^0 \wedge (V^0 \wedge H^0) \\ \\ \partial_t H^0 + \text{curl } E = -V^0 \wedge H^0 + V^0 \wedge (V^0 \wedge H^0) \\ \\ \partial_t E^0 - \text{curl } H^0 = 0 \end{cases} \tag{3.3}$$

For this system, the hypersurface $\mathbb{R} \times \partial\Omega$ is characteristic (of constant multiplicity). Hence, it admits classical piecewise regular (Sobolev) solutions discontinuous across $\mathbb{R} \times \partial\Omega$ ([26], [28], [29], [27]). More precisely, if $m \in \mathbb{N}$, and if we call $\Omega' := \mathbb{R}^3 \backslash \overline{\Omega}$, let us denote by $p\!-\!H^m(\Omega)$ the space of functions $v \in L^2(\mathbb{R}^3)$ such that $v_{|\Omega} \in H^m(\Omega)$ and $v_{|\Omega'} \in H^m(\Omega')$. The space $p\!-\!H^m(\Omega)$ is endowed with the natural norm $\|v_{|\Omega}\|_{H^m(\Omega)} + \|v_{|\Omega'}\|_{H^m(\Omega')}$. As before, we use the notation $p\!-\!\mathbf{H}^m(\Omega)$ when the function is valued in $\mathbb{R}^3$. For any given m, it is a consequence of the theory of discontinuous solutions of hyperbolic semilinear systems ([26], [28], [29], [27]) that the system (3.1) has solutions which satisfy

$$\overline{u^0}, E^0, H^0 \in \mathcal{C}^1\big([0,T], p\!-\!\mathbf{H}^m(\Omega) \big), \tag{3.4}$$

for some $T > 0$. For m big enough ($m > 3/2$), and inside this class of functions, it is equivalent to solve system (3.1) with initial datas

$$u^0_{|t=0} = u^0_0, \;\; E^0_{|t=0} = E^0_0, \;\; H^0_{|t=0} = H^0_0, \tag{3.5}$$

or to solve the system (3.3) with initial conditions

$$V^0_{|t=0} = \overline{u^0_0}, \;\; E^0_{|t=0} = E^0_0, \;\; H^0_{|t=0} = H^0_0. \tag{3.6}$$

In this paper we consider such solutions of system (3.1) which satisfy

$$\overline{u^0}, E^0, H^0 \in \mathcal{C}^1\big([0,T], p\!-\!\mathbf{H}^5(\Omega) \big), \tag{3.7}$$

for some $T > 0$. Our result in this section is that such a solution is actually the limit of a sequence of solutions of original system (1.1)-(1.2) as ε goes to zero. In order to state the result, let us introduce a function $\varphi \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R})$ such that $\Omega = \{\varphi > 0\}$, $\Omega' = \{\varphi < 0\}$, $\partial\Omega = \{\varphi = 0\}$ and normalized such that $|\nabla\varphi(x)| = 1$ for all x in a neighborood $\mathcal{V}$ of $\partial\Omega$. This implies that $\varphi(x) = \text{dist}(x, \partial\Omega)$ on $\mathcal{V} \cap \Omega$.

Theorem 3.1 *Assume that* (u^0, H^0, E^0) *satisfies the system* (3.1), (3.2) *and the condition* (3.7), *for some* $T > 0$. *Then the following holds.*

1. *There exists a family of initial datas* $(u_0^\varepsilon, H_0^\varepsilon, E_0^\varepsilon)_{\varepsilon>0}$ *satisfying the assumptions of theorem* 2.1 *such that the corresponding solution* $(u^\varepsilon, H^\varepsilon, E^\varepsilon)$ *of* $(1.1) - (1.2)$ *given by theorem* 2.1 *exists on* $[0, T]$ *and converges to* (u^0, H^0, E^0) *in* $\mathcal{C}\big([0, T], \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\mathbb{R}^3) \times \mathbf{L}^2(\mathbb{R}^3)\big)$, *as* $\varepsilon \to 0$.

2. *If* $u_0 := u^0_{|t=0}$ *satisfies* $\partial_{\mathbf{n}} u_{0|\partial\Omega} = 0$. *Then, the solution* $(u^\varepsilon, H^\varepsilon, E^\varepsilon)$ *of* $(1.1) - (1.2)$ *given by theorem* 2.1 *with initial data* $(u^0_{|t=0}, H^0_{|t=0}, E^0_{|t=0})$ *exists on* $[0, T]$ *and converges to* (u^0, H^0, E^0) *in* $\mathcal{C}\big([0, T], \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\mathbb{R}^3) \times \mathbf{L}^2(\mathbb{R}^3)\big)$, *as* $\varepsilon \to 0$.

3. *In both cases (1 et 2) there exists a boundary layer profile* $\mathbf{V}(t, x, z) \in \mathcal{C}([0, T], H^4(\Omega) \otimes H^4([0, +\infty[)$ *such that:*

$$\begin{cases} u^\varepsilon(t, x) = U^0(t, x) + \varepsilon \mathbf{V}(t, x, \dfrac{\varphi(x)}{\varepsilon}) + \varepsilon r^\varepsilon(t, x) \\ H^\varepsilon(t, x) = H^0(t, x) + \varepsilon R_H^\varepsilon(t, x) \\ E^\varepsilon(t, x) = E^0(t, x) + \varepsilon R_E^\varepsilon(t, x) \end{cases}$$

   with the following uniform estimate

$$\|r^\varepsilon\|_{L^\infty(0,T;H^1)} + \|\varepsilon r^\varepsilon\|_{L^\infty(0,T;H^2)} + \|R_H^\varepsilon\|_{L^\infty(0,T;\mathbf{H}_{\text{curl}})} + \|R_E^\varepsilon\|_{L^\infty(0,T;\mathbf{H}_{\text{curl}})} \;\leq\; C.$$

Note that in the point *1.* the function u^0 is not supposed to satisfy any boundary condition, and in particular the trace $u_0 := u^0_{|t=0}$ is *not supposed* to satisfy $\partial_{\mathbf{n}} u^0_{|[0,T]\times\partial\Omega} = 0$. This comment is to emphasize the fact that one cannot apply the existence theorem (2.1) with the initial values $u^0_{|t=0}, E^0_{|t=0}, H^0_{|t=0}$. On the other hand, this is a natural motivation for the point *2.*.

4 Reduction of the problem

4.1 The modified equation for u.

For regular solutions, the equation (1.1) is equivalent to the following equation (see [9]):

$$\partial_t u - \varepsilon^2 \Delta u - \varepsilon^2 u \wedge \Delta u = \varepsilon^2 |\nabla u|^2 u + u \wedge H - u \wedge (u \wedge H) \quad (\text{ in } \mathbb{R} \times \Omega) \quad . \tag{4.1}$$

Let us introduce some notations. We will note $\mathcal{P}_\perp$ the orthogonal projector of $L^2(\mathbb{R}^3; \mathbb{R}^3)$ onto the subspace of divergence free vector fields, and $\mathcal{P}_\| := \text{Id} - \mathcal{P}_\perp$. For convenience we will also use the notations $v_\perp := \mathcal{P}_\perp v$ and $v_\| := \mathcal{P}_\| v$ if $v \in L^2(\mathbb{R}^3;\mathbb{R}^3)$. Using the Fourier transform $\widehat{\ }$ in $L^2(\mathbb{R}^3;\mathbb{R}^3)$ gives the following expressions

$$\widehat{v_\|}(\xi) = |\xi|^{-2} \langle \xi . \widehat{v}(\xi)\rangle \, \xi \quad , \quad \widehat{v_\perp}(\xi) = -|\xi|^{-2} \, \xi \wedge (\xi \wedge \widehat{v}(\xi)) \tag{4.2}$$

where $\langle . \rangle$ is the scalar product and $\wedge$ the vectorial product in $\mathbb{R}^3$.

The relation (1.3) means that $\mathcal{P}_\|(H) + \mathcal{P}_\|(\bar{u}) = 0$. Replacing then $H = H_\perp - \mathcal{P}_\|(\bar{u})$ in the Landau-Lifschitz equation (4.1), gives the following equation

$$\begin{aligned} \partial_t u - \varepsilon^2 \ \Delta u = \varepsilon^2 \ \wedge \Delta u + \varepsilon^2 |\nabla u|^2 u \\ - u \wedge \mathcal{P}_\|(\bar{u}) + u \wedge \big(u \wedge \mathcal{P}_\|(\bar{u})\big) \\ + u \wedge H_\perp - u \wedge (u \wedge H_\perp) \quad . \end{aligned} \tag{4.3}$$

4.2 The wave equation for $H_\perp$.

In a classical way, we use the Maxwell system to get a scalar wave equation on $H_\perp$, with a right hand side depending on $\bar{u}$: we apply ∂_t to the first equation in (1.2) and take the curl of the second equation to get

$$\partial_t^2 H_\perp - \ \Delta H_\perp = -\partial_t^2 \mathcal{P}_\perp(\bar{u}). \tag{4.4}$$

We are then interested in solving the following non linear system of equations

$$\begin{aligned} \partial_t u - \varepsilon^2 \ \Delta u = \varepsilon^2 \ u \wedge \Delta u + \varepsilon^2 |\nabla u|^2 u \\ - u \wedge \mathcal{P}_\|(\bar{u}) + u \wedge \big(u \wedge \mathcal{P}_\|(\bar{u})\big) \\ + u \wedge \mathbf{h} - u \wedge (u \wedge \mathbf{h}) \quad \text{in }]0,\infty[\times\Omega, \end{aligned} \tag{4.5}$$

$$\partial_t^2 \mathbf{h} - \ \Delta \mathbf{h} = -\partial_t^2 \mathcal{P}_\perp(\bar{u}) \quad \text{in }]0,\infty[\times\mathbb{R}^3, \tag{4.6}$$

with boundary condition

$$\partial_{\mathbf{n}} u_{|]0,\infty[\times\partial\Omega} = 0 \tag{4.7}$$

and initial conditions for u

$$u_{|t=0} = u_0 \text{ in } \Omega, \tag{4.8}$$

and for $\mathbf{h}$

$$\mathbf{h}_{|t=0} = \mathbf{h}_0, \quad \partial_t \mathbf{h}_{|t=0} = \mathbf{h}_1 \quad \text{in } \mathbb{R}^3. \tag{4.9}$$

4.3 Initial data and compatibility conditions

A natural question is to express the initial data for $H_\perp$ and $\partial_t H_\perp$ in terms of the original data u_0, E_0, H_0. Concerning $H_\perp$ we just have:

$$(H_\perp)_{|t=0} = \mathcal{P}_\perp H_0. \tag{4.10}$$

For $\partial_t H_\perp$ we must use the equations. The Maxwell equations imply

$$\partial_t H_\perp = -\text{curl } E - \partial_t \mathcal{P}_\perp(\bar{u}). \tag{4.11}$$

The modified Landau-Lifschitz equation (4.5) writes

$$\partial_t u = \mathcal{F}\big(H_\perp, u, \nabla u, \Delta u, \mathcal{P}_\|(\bar{u})_{|\Omega}\big)$$

with obvious notations. Let us call

$$\begin{aligned}\mathbf{F}_0 &:= \Big(\mathcal{F}\big(H_\perp, u, \nabla u, \Delta u, \mathcal{P}_{\|}(\bar{u})_{|\Omega}\big)\Big)_{|t=0} \\ &= \mathcal{F}\big(\mathcal{P}_\perp H_0, u_0, \nabla u_0, \Delta u_0, \mathcal{P}_{\|}(\bar{u_0})_{|\Omega}\big).\end{aligned}$$

It follows that

$$\big(\partial_t \mathcal{P}_\perp(\bar{u})\big)_{|t=0} = \mathcal{P}_\perp(\overline{\mathbf{F}_0}).$$

Coming back to equation (4.11) we find the following expression for the initial value of $\partial_t H_\perp$, expressed with the original datas u_0, E_0, H_0:

$$(\partial_t H_\perp)_{t=0} = -\text{curl } E_0 - \mathcal{P}_\perp(\overline{\mathbf{F}_0}). \tag{4.12}$$

We will solve the wave equation for $H_\perp$ in the space

$$\mathcal{C}\big(0, T; H^1(\mathbb{R}^3)\big) \cap \mathcal{C}^1\big(0, T; L^2(\mathbb{R}^3)\big),$$

so we need an initial data for $\partial_t H_\perp$ in $\mathbf{L}^2(\mathbb{R}^3)$. This requirement will be our first "compatibility condition". Since $u_0 \in H^2(\Omega)$, we see that $\mathbf{F}_0$ and also $\mathcal{P}_\perp(\overline{\mathbf{F}_0})$ are in $L^2(\mathbb{R}^3)$. In view of relation(4.11) in follows that the condition $(\partial_t H_\perp)_{|t=0} \in L^2(\mathbb{R}^3)$ reduces to the following *necessary compatibility condition*

$$\text{curl } E_0 \in \mathbf{L}^2(\mathbb{R}^3). \tag{4.13}$$

This is the reason why we assume that our initial data E_0 belongs to $\mathbf{H}_{\text{curl}}$.
Let us turn now to the compatibility conditions for u_0. The point is that the function $\partial_t u$ has to be in $\mathcal{C}\big([0, T] : \mathbf{H}^1(\Omega)\big)$. A necessary compatibility condition is then

$$\nabla \mathbf{F}_0 \in \mathbf{L}^2(\Omega)\ . \tag{4.14}$$

This condition is always fulfilled when u_0 belongs to $\mathbf{H}^3(\Omega)$.

4.4 An existence result and the proof of theorem (2.1).

Let us first state the main theorem of this section.

Theorem 4.1 *Let $u_0 \in \mathbf{H}^3(\Omega)$ such that $\partial_{\mathbf{n}} u_{0|\partial\Omega} = 0$ and let $\mathbf{h}_0 \in \mathbf{H}^1(\mathbb{R}^3)$ and $\mathbf{h}_1 \in \mathbf{L}^2(\mathbb{R}^3)$. There exists $T > 0$ and a unique solution $(u, \mathbf{h})$ to the system (4.5)$\cdots$(4.9) on $]0, T[\times\Omega$ such that $u \in \mathcal{A}(T)$ and*

$$\mathbf{h} \in \mathcal{C}^1\big([0, T]; \mathbf{L}^2(\mathbb{R}^3)\big) \cap \mathcal{C}\big([0, T]; \mathbf{H}^1(\mathbb{R}^3)\big). \tag{4.15}$$

Moreover, if $\mathcal{P}_{\|}\mathbf{h}_j = 0$ for $j = 0, 1$,then $\mathcal{P}_{\|}\mathbf{h}(t, .) = 0$ for all $t \in [0, T]$.

Assuming for a moment theorem 4.1, we can now give the proof of theorem 2.1. Apply theorem 4.1 with initial datas u_0 and $\mathbf{h}_0 := \mathcal{P}_\perp(H_0)$ and

$$\mathbf{h}_1 := \text{curl } E_0 - \mathcal{P}_\perp(\overline{\mathbf{F}_0})\ \in \mathbf{L}^2(\mathbb{R}^3).$$

This gives a function $u \in \mathcal{A}(T)$ and a function $\mathbf{h}$, with regularity (4.15) satisfying $\mathcal{P}_{\|}\mathbf{h} = 0$, because of the last observation in theorem 4.1. Now, one can solve the Maxwell hyperbolic system (1.2) with $\partial_t \bar{u}$ in the right hand side

$$\begin{cases} \partial_t H + \text{curl } E = \partial_t \bar{u} \\ \partial_t E - \text{curl } H = 0 \\ (E, H)_{|t=0} = (E_0, H_0). \end{cases} \tag{4.16}$$

Since $\partial_t(\bar{u}) = \overline{(\partial_t u)}$ is in $\mathbf{L}^2([0,T] \times \mathbb{R}^3)$ it follows that this equation has a unique solution $(H,E) \in \mathcal{C}\big([0,T]; L^2(\mathbb{R}^3;\mathbb{R}^6)\big)$.
Let us consider now the wave equation (4.4) satisfied by $H_\perp$. Observing that the initial values $(H_\perp)_{|t=0}$ and $(\partial_t H \perp)_{|t=0}$ are the same as the initial data $\mathbf{h}_0$ and $\mathbf{h}_1$ (since of relation (4.12)), we deduce that $H_\perp = \mathbf{h}$. Now, the fact that H belongs to $\mathcal{C}\big([0,T];\mathbf{H}_{\text{curl}}\big)$ is a consequence of the following lemma.

Lemma 4.1 *A function v given in $\mathbf{L}^2(\mathbb{R}^3)$ belongs to $\mathbf{H}_{\text{curl}}$ if and only if $v_\perp$ is in $\mathbf{H}^1(\mathbb{R}^3)$. In such a case, it satisfies the inequality*

$$c^{-1}\ \|\text{curl }v\|_{L^2(\mathbb{R}^3)} \leq \|\nabla v_\perp\|_{L^2(\mathbb{R}^3)} \leq c\ \|\text{curl }v\|_{L^2(\mathbb{R}^3)} \quad ,$$

for some $c > 0$ independent of v.

Proof.
We write $v = v_\| + v_\perp$ and by Fourier transform on $\mathbb{R}^3$, $\widehat{(\text{curl }v)}(\xi) = i\xi \wedge \widehat{v}(\xi) = i\xi \wedge \widehat{v_\perp}(\xi)$. Now, since ξ and $v_\perp(\xi)$ are orthogonal vectors, noting $|.|$ the Euclidean norm in $\mathbb{R}^3$ we have: $|\widehat{(\text{curl }v)}(\xi)| = |\xi|\ |\widehat{v}(\xi)|$. Now, using the Parseval-Plancherel equality we obtain the lemma.

Then, as we already observed in section 2, we have

$$H_\| = -\mathcal{P}_\|(\bar{u}) \in \mathcal{C}^1\big([0,T];\mathbf{L}^2(\mathbb{R}^3)\big).$$

which implies that $H = H_\| + H_\perp$ is also $\mathcal{C}^1$ from $[0,T]$ to $\mathbf{L}^2(\mathbb{R}^3)$. Applying the time derivative ∂_t to the Maxwell system (4.16) we see that $H' := \partial_t H$ and $E' := \partial_t E$, are solutions of

$$\begin{cases} \partial_t H' + \text{curl }E' = \overline{\partial_t^2 u} \in L^2([0,T] \times \mathbb{R}^3;\mathbb{R}^3) \\ \partial_t E' - \text{curl }H' = 0 \\ (E',H')_{|t=0} = (-\text{curl }E_0 - \mathbf{F}_0, \text{curl }H_0) \in L^2(\mathbb{R}^3;\mathbb{R}^6)\ , \end{cases} \tag{4.17}$$

which implies that $\partial_t E$ (and also $\partial_t H$, which is already known) is in $\mathcal{C}\big([0,T];\mathbf{L}^2(\mathbb{R}^3)\big)$. It remains to prove that curl E belongs to $\mathcal{C}\big([0,T];\mathbf{L}^2(\mathbb{R}^3)\big)$. This follows from the first equation of the Maxwell system

$$\text{curl }E = \partial_t \bar{u} - \partial_t H$$

because of the regularity of H and u. This proves theorem 2.1.

The next section is devoted to the proof of theorem 4.1.

5 Proof of Theorem 4.1

For the proof of theorem 4.1 we use *a priori* estimates on a Galerkin approximation. The approximation space is based on the eigenspaces of the Laplacian on the domain

$$\mathcal{D}(\Delta) = \{u \in H^2(\Omega) \text{ such that } \partial_{\mathbf{n}} u_{|\partial\Omega} = 0\}.$$

Let's call Π_n the usual orthogonal projector on the finite dimensional invariant subspace built on the first n eigenspaces.

Our goal is to establish *a priori* estimates, uniform in n on the solution $(u_n, \mathbf{h}_n)$ of the following non linear problem (where we note simply $(u, \mathbf{h})$ instead of $(u_n, \mathbf{h}_n)$):

$$\begin{aligned}\partial_t u - \varepsilon^2\ \Delta u = \Pi_n\Big(\ \varepsilon^2\ \wedge \Delta u + \varepsilon^2 |\nabla u|^2 u\ \Big)\\ - \Pi_n\Big(\ u \wedge \mathcal{P}_{\|}(\bar{u}) + u \wedge \big(u \wedge \mathcal{P}_{\|}(\bar{u})\big)\ \Big)\\ + \Pi_n\Big(\ u \wedge \mathbf{h} - u \wedge (u \wedge \mathbf{h})\ \Big) \quad \text{in }]0,\infty[\times\Omega \quad ,\end{aligned} \tag{5.1}$$

$$\partial_t^2 \mathbf{h} -\ \Delta \mathbf{h} = -\partial_t^2 \mathcal{P}_{\perp}(\bar{u}) \quad \text{in }]0,\infty[\times\mathbb{R}^3 \quad , \tag{5.2}$$

with boundary condition

$$\partial_{\mathbf{n}} u_{|]0,\infty[\times\partial\Omega} = 0 \tag{5.3}$$

and initial conditions

$$u_{|t=0} = \Pi_n u_0 \text{ in } \Omega, \quad (\mathbf{h}, \partial_t \mathbf{h})_{|t=0} = (\mathbf{h}_0, \mathbf{h}_1) \text{ in } \mathbb{R}^3. \tag{5.4}$$

5.1 Technical lemmas and notations

For $m \geq 0$, We will note $H^m(\Omega) = W^{m,2}(\Omega)$ the usual Sobolev space, and we will note $\|.\|_m$ the usual norm

$$\|v\|_m := \sum_{|\alpha| \leq m} \|\partial_x^\alpha v\|_{L^2(\Omega)} \ .$$

We will denote by $\mathbf{H}^m(\Omega) := H^m(\Omega; \mathbb{R}^3)$ and will still denote $\|.\|_m$ the corresponding norm on $\mathbf{H}^m(\Omega)$. We will also use the corresponding notations with $\mathbb{R}^3$ in place of Ω. We will use many times the following lemma (see [1], [2], [31]).

Lemma 5.1 *Let Ω be a regular open subset of $\mathbb{R}^3$. On the linear space*

$$\mathbf{V} := \{u \in H^2(\Omega) \text{ such that } \partial_{\mathbf{n}} u_{|\partial\Omega} = 0\},$$

the norms $\|u\|_{H^2}$ and $\|u\|_{L^2} + \|\Delta u\|_{L^2}$ are equivalent. On the subspace $H^3(\Omega) \cap \mathbf{V}$, the norms $\|u\|_{H^3}$ and $\|u\|_{H^2} + \|\nabla \Delta u\|_{L^2}$ are equivalent.

The following result is also very useful in the study of ferromagnetism equations, and a proof can be found in [9] and [10].

Lemma 5.2 *Let $m \geq 0$ and $p \in]1,\infty[$. The mapping $u \to \big(\mathcal{P}_{\|}(\bar{u})\big)_{|\Omega}$ is continuous from $W^{m,p}(\Omega)$ into $W^{m,p}(\Omega)$. The same is true with $\mathcal{P}_{\perp}$.*

Let us also recall that $H^1(\Omega)$ is continuously embedded in $L^6(\Omega)$ ([1], [2]).

5.2 Estimates on h

Let us begin with the classical estimate for the wave equation, obtained by taking the scalar product of the equation with $\partial_t \mathbf{h}$.

We get:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\big(\|\partial_t \mathbf{h}\|_{L^2}^2 + \|\nabla \mathbf{h}\|_{L^2}^2\big) \ &\leq\ \|\mathcal{P}_{\perp}(\partial_t^2 \bar{u})\|_{L^2}\ \|\partial_t \mathbf{h}\|_{L^2}\\ &\leq\ \|\partial_t^2 u\|_{L^2}\ \|\partial_t \mathbf{h}\|_{L^2}.\end{aligned} \tag{5.5}$$

In order to get also an estimate on $\|\mathbf{h}\|_{L^2}$, we add to this estimate the obvious inequality $1/2\ \frac{d}{dt}(\|h\|^2_{L^2}) \leq \|h\|_{L^2}\|\partial_t h\|_{L^2}$. This gives the following energy inequality:

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{h}\|^2_{L^2} + \|\partial_t \mathbf{h}\|^2_{L^2} + \|\nabla \mathbf{h}\|^2_{L^2}\right) \ \leq\ \left(\|\mathbf{h}\|_{L^2} + \|\partial_t^2 u\|_{L^2}\right) \|\partial_t \mathbf{h}\|_{L^2}\ . \tag{5.6}$$

In view of the right hand side of this estimate, we are lead to look for estimates on *time derivatives of* u in order to control the term $\|\partial_t^2 u(s)\|_{L^2(\Omega)}$. This is an important difference with the "quasistatic case" as treated for example in [9] and [10].

5.3 Estimation on $\|u(t)\|_{L^2}$

Taking the scalar product of the equation with u, and integrating by parts gives

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2} + \varepsilon^2\|\nabla u(t)\|^2_{L^2} \leq \varepsilon^2\|u(t)\|^2_{L^\infty}\|\nabla u(t)\|^2_{L^2} \tag{5.7}$$

5.4 Estimation on $\|\nabla u(t)\|_{L^2}$

Let us write the equation (5.1) in the form:

$$\partial_t u - \varepsilon^2\Delta u = \varepsilon^2\Pi_n\Big(\chi(u)\wedge\Delta u\Big)\ + \Pi_n f\ , \tag{5.8}$$

where

$$f = \varepsilon^2|\nabla u|^2 u + u\wedge\big(\mathbf{h} - \mathcal{P}_{\|}(\bar u)\big) + u\wedge\big(u\wedge(\mathcal{P}_{\|}(\bar u) - \mathbf{h})\big)\ .$$

Let us form the scalar product of the equation and Δu, in $L^2(\Omega)$. Integrating once by parts we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2_{L^2} + \varepsilon^2\|\Delta u(t)\|^2_{L^2} \leq \|f\|_{L^2}\ \|\Delta u(t)\|_{L^2}\ . \tag{5.9}$$

We control $\|f\|_{L^2}$ as follows.

$$\begin{aligned}\|f\|_{L^2}\ &\leq\ \varepsilon^2\||\nabla u|^2 u\|_{L^2} + \|u\wedge\big(\mathbf{h} - \mathcal{P}_{\|}(\bar u)\big)\|_{L^2}\\ &\qquad + \|u\wedge\big(u\wedge(\mathcal{P}_{\|}(\bar u) - \mathbf{h})\big)\|_{L^2}\\ &\leq\ \varepsilon^2\ \|u(t)\|_{L^\infty}\|\nabla u(t)\|_{L^\infty}\|\nabla u(t)\|_{L^2}\\ &\qquad + \big(\ \|u(t)\|_{L^\infty} + \|u(t)\|^2_{L^\infty}\ \big)\big(\ \|u(t)\|_{L^2} + \|\mathbf{h}(t)\|_{L^2}\ \big)\ .\end{aligned} \tag{5.10}$$

By Sobolev embedding of $H^2(\Omega)$ in $L^\infty(\Omega)$ we get

$$\begin{aligned}\|f(t)\|_{L^2} \leq\ &\ \varepsilon^2\|u(t)\|^2_{H^2}\ \|u(t)\|_{H^3}\\ &+ c\ \big(\|u(t)\|_{H^2} + \|u(t)\|^2_{H^2}\big)\ \big(\|u(t)\|_{L^2} + \|\mathbf{h}(t)\|_{L^2}\big).\end{aligned} \tag{5.11}$$

We obtain then the following estimate:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\ \|\nabla u(t)\|^2_{L^2} + \varepsilon^2\|\Delta u(t)\|^2_{L^2} &\leq c\ \|u(t)\|^3_{H^2}\ \|u(t)\|_{H^3}\\ &\quad c\ \big(\ \|u(t)\|^2_{H^2} + \|u(t)\|^3_{H^2}\ \big)\ \big(\ \|u(t)\|_{L^2} + \|\mathbf{h}(t)\|_{L^2}\ \big).\end{aligned} \tag{5.12}$$

5.5 Estimation on $\|\Delta u(t)\|_{L^2}$

Taking the scalar product of the equation 5.8 with $\Delta^2 u$, and integrating by part. We obtain the inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta u\|_{L^2}^2+\varepsilon^2\ \|\nabla\Delta u\|_{L^2}^2\leq\ &\varepsilon^2\ \|\nabla(u)\wedge\Delta u\|_{L^2}\,\|\nabla\Delta u\|_{L^2}+\|\nabla f\|_{L^2}\|\nabla\Delta u\|_{L^2}\\ \leq\ &c\ \|\,|\nabla u|\,|\Delta u|\,\|_{L^2}\,\|u\|_{H^3}+\|\nabla f\|_{L^2}\|u\|_{H^3}\\ \leq\ &c\ \|\nabla u\|_{L^6}\ \|\Delta u\|_{L^3}\ \|u\|_{H^3}+\|\nabla f\|_{L^2}\ \|u\|_{H^3}\end{aligned}\tag{5.13}$$

with a constant c independant of u.

Let us recall the Sobolev embedding

$$\|u\|_{L^6(\Omega)}\ \leq\ c\ \|u\|_{H^1(\Omega)}\ .\tag{5.14}$$

By interpolation betwen L^2 and L^6 we deduce from (5.14) the inequality

$$\|u\|_{L^3(\Omega)}\ \leq\ c\ \|u\|_{L^2}^{1/2}\ \|u\|_{H^1(\Omega)}^{1/2}\ .\tag{5.15}$$

Using (5.14), (5.15) we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta u\|_{L^2}^2+\varepsilon^2\ \|\nabla\Delta u\|_{L^2}^2\ \leq\ c\ \|u\|_{H^2}^{3/2}\ \ \|u\|_{H^3}^{3/2}+\|\nabla f\|_{L^2}\ \|u\|_3\tag{5.16}$$

We estimate $\|\nabla f\|_{L^2}$ in the following way.

$$\begin{aligned}\|\nabla f\|_{L^2}\ \leq\ &\varepsilon^2\|\ |\nabla u|^3\ \|_{L^2}+\varepsilon^2\|\ |u|\ |\nabla u|\ |D^2u|\ \|_{L^2}\\ &+\|\ |\nabla u|\ |\mathcal{P}_{\|}(u)-\mathbf{h}|\ \|_{L^2}+\|\ |u|\ |\nabla\big(\mathcal{P}_{\|}(\bar u)\big)-\nabla\mathbf{h}|\ \|_{L^2}\\ &+\||u||\nabla u||\mathcal{P}_{\|}(\bar u)-\mathbf{h}|\|_{L^2}+\|\ |u|^2\ |\nabla\big(\mathcal{P}_{\|}(\bar u)\big)-\nabla\mathbf{h}|\ \|_{L^2}\\ \leq\ &\varepsilon^2\ \|\nabla u\|_{L^6}^3+\varepsilon^2\ \|u\|_{L^\infty}\ \|\nabla u\|_{L^3}\ \|D^2u\|_{L^6}\\ &+\|\nabla u\|_{L^3}\ (\|u\|_{L^6}+\|\mathbf{h}\|_{L^6})+\|u\|_{L^\infty}\ (\|u\|_{H^1}+\|\mathbf{h}\|_{H^1})\\ &+\|u\|_{L^\infty}\ \|\nabla u\|_{L^3}\ (\|u\|_{L^6}+\|\mathbf{h}\|_{L^6})+\|u\|_{L^\infty}^2\ (\|u\|_{H^1}+\|\mathbf{h}\|_{H^1})\ .\end{aligned}\tag{5.17}$$

Now, using again the inequalities (5.14) and (5.15), we obtain

$$\begin{aligned}\|\nabla f\|_{L^2}\ \leq\ &\varepsilon^2\ c\ \|u\|_{H^2}^3+c\ \|u\|_{L^\infty}\ \|u\|_{H^2}^{3/2}\|u\|_{H^3}^{1/2}\\ &+c\ \|u\|_{H^2}\ (\|u\|_{H^1}+\|\mathbf{h}\|_{H^1})+\|u\|_{L^\infty}\ (\|u\|_{H^1}+\|\mathbf{h}\|_{H^1})\\ &+c\ \|u\|_{H^2}\|u\|_{L^\infty}\ (\|u\|_{H^1}+\|\mathbf{h}\|_{H^1})+\|u\|_{L^\infty}^2\ (\|u\|_{H^1}+\|\mathbf{h}\|_{H^1})\ .\end{aligned}\tag{5.18}$$

Using then the Sobolev embedding of $H^2(\Omega)$ in $L^\infty(\Omega)$, we obtain

$$\begin{aligned}\|\nabla f\|_{L^2}\ \leq\ &c\ (\|u\|_{H^2}^2+\|u\|_{H^2}^3)+c\ (\|u\|_{H^2}+\|u\|_{H^2}^2)\ \|\mathbf{h}\|_{H^1}\\ &+c\ \|u\|_{H^2}^{5/2}\ \|u\|_{H^3}^{1/2}\ .\end{aligned}\tag{5.19}$$

We have then the following estimate:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Delta u\|_{L^2}^2+\varepsilon^2\ \|\nabla\Delta u\|_{L^2}^2\ \leq\\ &\quad c\ \|u\|_{H^2}^{3/2}\ \ \|u\|_{H^3}^{3/2}+c\ (\|u\|_{H^2}^2+\|u\|_{H^2}^3)\ \|u\|_{H^3}\\ &\quad+c\ (\|u\|_{H^2}+\|u\|_{H^2}^2)\ \|\mathbf{h}\|_{H^1}\|u\|_{H^3}+c\ \|u\|_{H^2}^{5/2}\ \|u\|_{H^3}^{3/2}\ .\end{aligned}\tag{5.20}$$

5.6 Estimations on $\partial_t u$

Applying ∂_t to the equation (5.1) we obtain

$$
\begin{aligned}
\partial_t^2 u - \varepsilon^2 \ \Delta\partial_t u = \varepsilon^2 \Pi_n\Big(\partial_t u \wedge \Delta u + u \wedge \partial_t \Delta u\ \Big) \\
+ \ \varepsilon^2 \Pi_n\Big(\ \ 2(\nabla u.\nabla \partial_t u)u \ + \ |\nabla u|^2 \partial_t u\ \Big) \\
- \ \Pi_n\Big(\ \partial_t u \wedge \mathcal{P}_{\|}(\bar{u}) + \ u \wedge \mathcal{P}_{\|}(\partial_t \bar{u})\Big) \\
+ \ \Pi_n\Big(\ \partial_t u \wedge \big(u \wedge \mathcal{P}_{\|}(\bar{u})\big) + u \wedge \big(\partial_t u \wedge \mathcal{P}_{\|}(\bar{u})\big) \\
+ u \wedge \big(u \wedge \mathcal{P}_{\|}(\partial_t \bar{u})\big)\ \Big) + \Pi_n\Big(\ \partial_t u \wedge \mathbf{h} + u \wedge \partial_t \mathbf{h}\ \Big) \\
- \ \Pi_n\Big(\ \partial_t u \wedge (u \wedge \mathbf{h}) - \ u \wedge (\partial_t u \wedge \mathbf{h}) \ - u \wedge (u \wedge \partial_t \mathbf{h})\ \Big)
\end{aligned} \tag{5.21}
$$

with boundary condition

$$\partial_{\mathbf{n}}(\partial_t u)_{|]0,\infty[\times\partial\Omega} = 0. \tag{5.22}$$

This equation has the form

$$\partial_t^2 u - \varepsilon^2 \Delta\partial_t u = \varepsilon^2 \Pi_n(u \wedge \Delta\partial_t u) + \Pi_n g \tag{5.23}$$

where g does not contain the term $\Delta\partial_t u$.

Taking the scalar product of the equation with $\partial_t u$, and performing the usual inegrations by parts, gives the following inequality :

$$\frac{1}{2}\frac{d}{dt}\big(\|\partial_t u\|_{L^2}^2\big) + \varepsilon^2 \ \|\partial_t \nabla u\|_{L^2}^2 \leq + \ c\varepsilon^2 \ \|\partial_t u\|_{L^2} \ \|\Delta\partial_t u\|_{L^2} \ + \ \|\partial_t u\|_{L^2} \ \|g\|_{L^2}. \tag{5.24}$$

Taking the scalar product of the equation with $\Delta\partial_t u$ and using one integration by parts, gives the estimate

$$\frac{d}{dt}\big(\|\nabla\partial_t u(t)\|_{L^2}^2\big) + \varepsilon^2 \ \|\Delta\partial_t u(t)\|_{L^2}^2 \ \leq \|g(t)\|_{L^2} \ \|\Delta\partial_t u\|_{L^2} \ . \tag{5.25}$$

Taking the scalar product of the equation with $\partial_t^2 u$ we obtain in the same way, the following inequality:

$$
\begin{aligned}
\varepsilon^2 \frac{d}{dt}\big(\|\nabla\partial_t u(t)\|_{L^2}^2\big) + \|\partial_t^2 u(t)\|_{L^2}^2 \ \leq \\
\varepsilon^2 c\,\|\Delta\partial_t u(t)\|_{L^2} \ \|\partial_t^2 u(t)\|_{L^2} \ + \ \|g(t)\|_{L^2} \ \|\partial_t^2 u(t)\|_{L^2} \ .
\end{aligned} \tag{5.26}
$$

Now, we control $\|g(t)\|_{L^2}$ in the following way.

$$
\begin{aligned}
\|g(t)\|_{L^2(\Omega)} \leq\ & \varepsilon^2\|\partial_t u\ \Delta u\|_{L^2} + 2\varepsilon^2\ \|u\,\nabla u\,\nabla\partial_t u\|_{L^2} \\
& + \varepsilon^2\|\partial_t u\,|\nabla u|^2\|_{L^2}\ +\ \|\partial_t u\,\mathcal{P}_{\|}(\bar u)\|_{L^2}\ +\ \|u\,\mathcal{P}_{\|}(\partial_t\bar u)\|_{L^2} \\
& + 2\|u\,\partial_t u\,\mathcal{P}_{\|}(\bar u)\|_{L^2}\ + \||u|^2\,\mathcal{P}_{\|}(\partial_t\bar u)\|_{L^2}\ +\ \|\partial_t u\,h\|_{L^2} \\
& + \|\partial_t u\,\mathbf{h}\|_{L^2}\ +\ \|u\,\partial_t\mathbf{h}\|_{L^2} +\ 2\|u\,\partial_t u\,\mathbf{h}\|_{L^2}\ + \||u|^2\,\partial_t\mathbf{h}\|_{L^2} \\
\leq\ & \varepsilon^2\|\partial_t u\|_{L^6}\|\Delta u\|_{L^3}\ + 2\varepsilon^2\|u\|_{L^\infty}\|\nabla u\|_{L^6}\|\nabla\partial_t u\|_{L^3} \\
& + \varepsilon^2\|\partial_t u\|_{L^6}\|\nabla u\|^2_{L^6}\ +\ c\ (1+2\|u\|_{L^\infty})\|\partial_t u\|_{L^6}\|u\|_{L^3} \\
& + c\ (\|u\|_{L^\infty} + \|u\|^2_{L^\infty})\|\partial_t u\|_{L^2} +\ (\|u\|_{L^\infty} + \|u\|^2_{L^\infty})\|\partial_t\mathbf{h}\|_{L^2} \\
& + (1+2\|u\|_{L^\infty})\|\partial_t u\|_{L^6}\|\mathbf{h}\|_{L^3}.
\end{aligned}
\tag{5.27}
$$

We obtain the following estimate, with a knew constant c:

$$
\begin{aligned}
\|g(t)\|_{L^2(\Omega)} \leq\ & \varepsilon^2 c\|\partial_t u\|_{H^1}\|u\|^{1/2}_{H^2}\|u\|^{1/2}_{H^3} + c\|u\|^2_{H^2}\|\partial_t u\|^{1/2}_{H^1}\|\partial_t u\|^{1/2}_{H^2} \\
& + \varepsilon^2 c\|\partial_t u\|_{H^1}\|u\|^2_{H^2} + c(1+2\|u\|_{H^2})\,\|\partial_t u\|_{H^1}\|u\|_{H^1} \\
& + c(\|u\|_{H^2} + \|u\|^2_{H^2})\,\|\partial_t u\|_{L^2} + c(\|u\|_{H^2} + \|u\|^2_{H^2})\|\partial_t\mathbf{h}\|_{L^2} \\
& + c(1+2\|u\|_{H^2})\,\|\partial_t u\|_{H^1}\|\mathbf{h}\|_{H^1}\ .
\end{aligned}
\tag{5.28}
$$

Now, adding inequalities (5.24) + (5.25)+ $\dfrac{1}{\lambda}$ (5.26), and chosing λ big enough, we "absorb" in left hand side the term $\|\partial_t^2 u\|_{L^2}\|\Delta\partial_t u\|_{L^2}$. We also absorb the term $\|\Delta\partial_t u\|_{L^2}$ in factor of $\|\partial_t u\|_{L^2}$, and we obtain the following estimate:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\partial_t u\|^2_{L^2} + 2(1+\frac{\varepsilon^2}{\lambda})\|\nabla\partial_t u\|^2_{L^2}\ \Big) + \frac{\varepsilon^2}{4}\|\Delta\partial_t u\|^2_{L^2} + \frac{1}{2\lambda}\|\partial_t^2 u\|^2_{L^2}\ \leq \\
&c^2\varepsilon^2\|\partial_t u\|^2_{L^2} + \left(\|\partial_t u\|_{L^2} + \|\Delta\partial_t u\|_{L^2} + \frac{1}{\lambda}\|\partial_t^2 u\|_{L^2}\right)\|g\|_{L^2}\ .
\end{aligned}
\tag{5.29}
$$

which is satisfied for any $\lambda \geq \lambda_0$ with a λ_0 big enough.

5.7 End of the proof

Recall that $u = u_n$, and $\mathbf{h} = \mathbf{h}_n$ Let us call $Q(t) = Q_n$ the quantity

$$
\begin{aligned}
Q(t) := \|u(t)\|^2_{L^2} + \|\nabla u(t)\|^2_{L^2} + \|\Delta u(t)\|^2_{L^2} + \\
+ \|\partial_t u\|^2_{L^2} + 2(1+\frac{\varepsilon^2}{\lambda})\|\nabla\partial_t u\|^2_{L^2} + \|\mathbf{h}\|^2_{H^1}\ .
\end{aligned}
\tag{5.30}
$$

Adding the previous estimates we derive the following inequality, (to simplify, we have written Q, u and $\mathbf{h}$, in place of $Q_n(t)$, $u_n(t)$ and $\mathbf{h}_n(t)$):

$$\begin{aligned} \frac{1}{2}\frac{dQ}{dt} &+ c\,(\|u\|_{H^3} + \|\Delta\partial_t u\|_{H^2} + \|\partial_t^2 u\|_{L^2})^2 \;\le \\ &A(Q) + B(Q)(\|u\|_{H^3}^2 + \|\Delta\partial_t u\|_{H^2} + \|\partial_t^2 u\|_{L^2}). \end{aligned} \tag{5.31}$$

where A, B are some polynomial functions, c is some positive constant, all *independent of* n. Absorbing in the left hand side the term

$$(\|u\|_{H^3} + \|\Delta\partial_t u\|_{H^2} + \|\partial_t^2 u\|_{L^2})^2$$

and noting $F = 2A + B$, we obtain the inequality

$$\frac{dQ}{dt} \;+\; c\,(\|u\|_{H^3} + \|\Delta\partial_t u\|_{H^2} + \|\partial_t^2 u\|_{L^2})^2 \;\le\; F(Q). \tag{5.32}$$

It remains to control that the family of initial values $Q_n(0)$ is uniformly bounded with respect to n. Here is the place where the compatibility conditions appear. Because of the regularity of the initial datas u_0 and $\mathbf{h}_0$, the quantity $Q_n(0)$ is uniformly bounded if and only if $\|\nabla\partial_t u_n(0)\|_{L^2}$ is uniformly bounded. The equation (5.1) implies that

$$(\partial_t u_n)_{|t=0} = \Pi_n \mathbb{F}^{\,0,n}\;, \tag{5.33}$$

with

$$\begin{aligned} \mathbb{F}_\chi^{\,0,n} := \;& \varepsilon^2\,\Delta u_{0,n} + \;\varepsilon^2 u_{0,n}\wedge\Delta u_{0,n} + \varepsilon^2|\nabla u_{0,n}|^2 u_{0,n} \\ & - u_{0,n}\wedge\mathcal{P}_{\|}(\overline{u_{0,n}}) + u_{0,n}\wedge\big(u_{0,n}\wedge\mathcal{P}_{\|}(\overline{u_{0,n}})\big) \\ & +\; u_{0,n}\wedge\mathbf{h}_{0,n} - u_{0,n}\wedge(u_{0,n}\wedge\mathbf{h}_{0,n})\;. \end{aligned} \tag{5.34}$$

We know that $\mathbf{h}_0 \in \mathbf{H}^1(\mathbb{R}^3)$ and $u_0 \in \mathbf{H}^3(\Omega)$. This implies that $\mathbf{h}_{0,n}$ and $u_{0,n}$ are respectively uniformly bounded in $\mathbf{H}^1(\mathbb{R}^3)$ and $\mathbf{H}^3(\Omega)$, which also implies that $u_{0,n}$ and $\nabla u_{0,n}$ are bounded in $L^\infty(\Omega)$. It follows that $\mathbb{F}_\chi^{0,n}$ is uniformly bounded in $H^1(\mathbb{R}^3)$, which implies that $Q_n(0)$ is uniformly bounded. The theorem 4.1 is now a classical consequence of the estimate (5.32).

6 BKW method for Theorem 3.1

The aim of this section is to work out the limit as ε goes to zero of the solution of (1.1). To perform this result we bring to the fore a small amplitude boundary layer induced by the Neuman boundary condition. The analysis follows the usual steps: first we construct an approximate solution by a BKW type analysis of the boundary layer, and second we justify the this asymptotic expansion, proving at the same time the existence of the exact solution and the asymptotic expansion.

6.1 Formal asymptotic expansion

We first recall the reduced system we have to study, and we introduce some notations.

$$
\begin{array}{ll}
(1) & \partial_t u^\varepsilon - \varepsilon^2 \Delta u^\varepsilon = \varepsilon^2 u^\varepsilon |\nabla u^\varepsilon|^2 + \varepsilon^2 u^\varepsilon \wedge \Delta u^\varepsilon - u^\varepsilon \wedge \mathcal{H}^\varepsilon \\
& \qquad + u^\varepsilon \wedge (u^\varepsilon \wedge \mathcal{H}^\varepsilon) - u^\varepsilon \wedge \mathbf{h}^\varepsilon + u^\varepsilon \wedge (u^\varepsilon \wedge \mathbf{h}^\varepsilon) \text{ in } \Omega, \\
(2) & \partial_{\mathbf{n}} u^\varepsilon = 0 \text{ on } \partial\Omega \\
(3) & \partial_t^2 \mathbf{h}^\varepsilon - \Delta \mathbf{h}^\varepsilon = -\partial_t^2 P_\perp(\bar{u^\varepsilon}) = -\partial_t^2(\bar{u^\varepsilon} + \mathcal{H}^\varepsilon) \text{ in } \mathbb{R}^3 \\
(4) & \text{div } (u^\varepsilon + \mathcal{H}^\varepsilon) = 0 \text{ in } \Omega, \quad \text{div } \mathcal{H}^\varepsilon = 0 \text{ in } {}^c\Omega \\
(5) & \text{curl } \mathcal{H}^\varepsilon = 0 \text{ in } \mathbb{R}^3.
\end{array}
\tag{6.1}
$$

Let us recall that the function $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$ satisfies

$$
\Omega = \{x,\ \varphi(x) > 0\},\ \partial\Omega = \{x,\ \varphi(x) = 0\}
$$

and $|\nabla\varphi(x)| \equiv 1$ in a neighbourhood $\mathcal{V}$ of $\partial\Omega$. With this definition $\nabla\varphi(x)$ define the inward unitary normal at the point $x \in \partial\Omega$, and ∂_n extends to all $\mathbb{R}^3$ as the vector field $\partial_{\mathbf{n}} \equiv -\sum(\partial_j\varphi)\,\partial_j$. In particular, $\partial_{\mathbf{n}}\varphi = -1$ on $\mathcal{V}$.
In the spirit of BKW method, we seek u^ε on the following form:

$$
u^\varepsilon(t,x) = U^0(t,x,\frac{\varphi(x)}{\varepsilon}) + \varepsilon U^1(t,x,\frac{\varphi(x)}{\varepsilon}) + \cdots .
$$

We split $U^i(t,x,z)$ as $\overline{U^i}(t,x) + \widetilde{U^i}(t,x,z)$, where $\overline{U^i}(t,x,z) = \lim_{z\to+\infty} U^i(t,x,z)$. Moreover we suppose that for any $\alpha \in \mathbb{N}^5$, $\lim\limits_{z\to+\infty} \partial^\alpha \widetilde{U^i}(t,x,z) = 0$. To be more precise we distinguish $\mathcal{H}^\varepsilon_{int} = \mathcal{H}^\varepsilon_{|\Omega}$ from $\mathcal{H}^\varepsilon_{ext} = \mathcal{H}^\varepsilon_{|{}^c\Omega}$ and we write

$$
\mathcal{H}^\varepsilon_{int}(t,x) = \mathcal{H}^0_{int}(t,x,\frac{\varphi(x)}{\varepsilon}) + \varepsilon\mathcal{H}^1_{int}(t,x,\frac{\varphi(x)}{\varepsilon}) + \cdots ,
$$

and

$$
\mathcal{H}^\varepsilon_{ext}(t,x) = \mathcal{H}^0_{ext}(t,x) + \varepsilon\mathcal{H}^1_{ext}(t,x) + \cdots ,
$$

that is there is no boudary layer outside Ω (this fact may be shown by formal expansion but for sake of simplicity we suppose it *a priori*).

The transmission conditions on $\mathcal{H}$ read

$$
\begin{aligned}
[\mathcal{H}\cdot\mathbf{n}] &= u\cdot\mathbf{n} \text{ on } \partial\Omega, \\
[\mathcal{H}\wedge\mathbf{n}] &= 0 \text{ on } \partial\Omega,
\end{aligned}
\tag{6.2}
$$

where we denote by $[f]$ the jump of f across $\partial\Omega$.

In the same way we write

$$
\mathbf{h}_{int} = \mathbf{h}^0_{int}(t,x,\frac{\varphi(x)}{\varepsilon}) + \varepsilon\mathbf{h}^1_{int}(t,x,\frac{\varphi(x)}{\varepsilon}) + \ldots ,
$$

and

$$
\mathbf{h}_{ext}(t,x) = \mathbf{h}^0_{ext}(t,x) + \varepsilon\mathbf{h}^1_{ext}(t,x) + \ldots ,
$$

with the transmission condition

$$[\mathbf{h}] = 0 \text{ on } \partial\Omega. \tag{6.3}$$

Order -2 From equation (3) in (6.1) we deduce that $h^0_{int,zz} = 0$ that is as $\lim_{z\to+\infty} \widetilde{h^0_{int}}(t,x,z) = 0$

$$\widetilde{h^0_{int}} = 0, \tag{6.4}$$

Order -1 The boundary condition (2) in (6.1) yields

$$\widetilde{U^0_z} = 0 \text{ at } z = 0, \text{ for } x \in \partial\Omega$$

Now, from (4) in (6.1)

$$\nabla\varphi \cdot \left(\mathcal{H}^0_{int,z} + U^0_z\right) = 0,$$

$$\nabla\varphi \wedge \mathcal{H}^0_{int,z} = 0.$$

¿From these two previous equation, we deduce that

$$\widetilde{\mathcal{H}^0_{int}} = -(\widetilde{U^0} \cdot \mathbf{n})\mathbf{n} \tag{6.5}$$

The equation (3) in (6.1) gives $\mathbf{h}^1_{int,zz} = 0$, and so

$$\widetilde{\mathbf{h}^1_{int}} = 0. \tag{6.6}$$

Order 0 We now write (1) in Equation (6.1) at the order 0 and we obtain

$$\begin{aligned} \partial_t U^0 - U^0_{zz} = \;& |U^0_z| U^0 + U^0 \wedge U^0_{zz} + U^0 \wedge (\mathcal{H}^0_{int} + \mathbf{h}^0_{int}) \\ & - U^0 \wedge (U^0 \wedge (\mathcal{H}^0_{int} + \mathbf{h}^0_{int})) \end{aligned} \tag{6.7}$$

and

$$\partial_{\mathbf{n}} U^0 + \partial_{\mathbf{n}}\varphi U^1_z = 0 \tag{6.8}$$

from the boundary conditions. To obtain the equation satisfied by $\overline{U^0}$ we perform the limit as $z \to +\infty$ in the above equation, and we find:

$$\partial_t \overline{U^0} = \overline{U^0} \wedge \left(\overline{\mathcal{H}^0_{int}} + \overline{\mathbf{h}^0_{int}}\right) - \overline{U^0} \wedge \left(\overline{U^0} \wedge \left(\overline{\mathcal{H}^0_{int}} + \overline{\mathbf{h}^0_{int}}\right)\right) \tag{6.9}$$

Substracting the previous equation from (6.7), we obtain as $\widetilde{\mathbf{h}^0_{int}} = 0$:

$$\begin{aligned} \partial_t \widetilde{U^0} - \widetilde{U^0_{zz}} =& |\widetilde{U^0_z}|^2 \left(\widetilde{U^0} + \overline{U^0}\right) + \left(\widetilde{U^0} + \overline{U^0}\right) \wedge U^0_{zz} \\ & + \widetilde{U^0} \wedge \left(\overline{\mathcal{H}^0_{int}} + \overline{\mathbf{h}^0_{int}}\right) + \overline{U^0} \wedge \widetilde{\mathcal{H}^0_{int}} + \widetilde{U^0} \wedge \widetilde{\mathcal{H}^0_{int}} + \text{ trilinear terms} \end{aligned}$$

¿From (6.5) the solution $\widetilde{U^0} = 0$ solves this equation, and by uniqueness argument one obtain:

$$\widetilde{U^0} = 0, \; \widetilde{\mathcal{H}^0_{int}} = 0. \tag{6.10}$$

¿From (4) in (6.1) we have

$$\begin{aligned}
&\operatorname{div} \mathcal{H}^0_{ext} = 0,\\
&\operatorname{div} \left(\mathcal{H}^0_{int} + \bar{U}^0\right) + \nabla\varphi \cdot \left(U^1_z + \mathcal{H}^1_{int,z}\right) = 0,\\
&\operatorname{curl} \mathcal{H}^0_{int} + \nabla\varphi \wedge \mathcal{H}^1_z = 0,\\
&\operatorname{curl} \mathcal{H}^0_{ext} = 0.
\end{aligned}$$

We can now derive the equation satisfied by $\mathcal{H}^0_{ext}$ and $\overline{\mathcal{H}^0_{int}}$

$$\operatorname{div} \mathcal{H}^0_{ext} = 0, \qquad \operatorname{curl} \mathcal{H}^0_{ext} = 0$$

$$\operatorname{div} \left(\overline{\mathcal{H}^0_{int}} + \overline{\bar{U}^0}\right) = 0, \qquad \operatorname{curl} \overline{\mathcal{H}^0_{int}} = 0,$$

The transmission conditions follow from (6.2) as according to (6.10) one has $\widetilde{\mathcal{H}^0_{int}} = 0$.

$$\begin{aligned}
\mathcal{H}^0_{ext} \wedge \mathbf{n} &= \overline{\mathcal{H}^0_{int}} \wedge \mathbf{n}\\
\mathcal{H}^0_{ext} \cdot \mathbf{n} &= \left(\overline{\mathcal{H}^0_{int}} + \overline{U^0}\right) \cdot n
\end{aligned}$$

¿From these last equation we deduce that:

$$\begin{aligned}
\overline{\mathcal{H}^0_{int}} &= P_{\|}(\overline{U^0})_{|\Omega}\\
\mathcal{H}^0_{ext} &= P_{\|}(\overline{U^0})_{|\Omega}
\end{aligned} \tag{6.11}$$

The equation (3) in (6.1) gives

$$\begin{aligned}
&\partial_t^2 \mathbf{h}^0_{int} - \Delta \mathbf{h}^0_{int} - \mathbf{h}^2_{int,zz} = -\partial_t^2 \left(U^0 + \mathcal{H}^0_{int}\right)\\
&\partial_t^2 \mathbf{h}^0_{ext} - \Delta \mathbf{h}^0_{ext} = \partial_t^2 \mathcal{H}^0_{ext}
\end{aligned}$$

Taking the limit as z goes to infinity, we obtain the equation satisfied by $\overline{\mathbf{h}^0}$:

$$\partial_t^2 \overline{\mathbf{h}^0} - \Delta \overline{\mathbf{h}^0} = -\partial_t^2 \left(\overline{\bar{U}^0} + \mathcal{H}^0\right),$$

that is,

$$\partial_t^2 \overline{\mathbf{h}^0} - \Delta \overline{\mathbf{h}^0} = -\partial_t^2 P_{\perp}(\overline{U^0}) \tag{6.12}$$

¿From the transmission condition (6.2), one has

$$\nabla\varphi \cdot \left(\mathcal{H}^1_{int,z} + U^1_z\right)$$

$$\nabla\varphi \wedge \mathcal{H}^1_{int,z} = 0.$$

These previous equations yields to

$$\widetilde{\mathcal{H}}^1_{int} = -(\widetilde{U^1} \cdot \mathbf{n})\mathbf{n} \tag{6.13}$$

Order 1

The equation (6.1) gives

$$\partial_t U^1 - U^1_{zz} = \quad U^0 \wedge U^1_{zz} + U^0 \wedge \left(\mathcal{H}^1 + h^1\right) + U^1 \wedge \left(\mathcal{H}^0 + \mathbf{h}^0\right)$$

$$+ \text{ trilinear terms}$$

$$\partial_{\mathbf{n}} U^1 + \partial_{\mathbf{n}} \varphi U^2_z = 0.$$

As already done, we obtain the equation satisfied by $\overline{U^1}$ performing the limit as $z \to +\infty$, and we find:

$$\partial_t \overline{U^1} = \overline{U^0} \wedge \left(\overline{\mathcal{H}^1_{ext}} + \overline{\mathbf{h}^1_{ext}}\right) + \overline{U^1} \wedge \left(\overline{\mathcal{H}^0_{ext}} + \overline{\mathbf{h}^0_{ext}}\right)$$

$$+ \text{ trilinear terms}$$

By difference we can write the equation satisfied by $\widetilde{U^1}$:

$$\partial_t \widetilde{U^1} - \widetilde{U^1}_{zz} = \overline{U^0} \wedge \widetilde{U^1}_{zz} + \overline{U^0} \wedge \widetilde{\mathcal{H}^1_{int}} + \widetilde{U^1} \wedge \left(\overline{\mathcal{H}^0} + \mathbf{h}^0\right)$$

$$+ \text{ trilinear terms}$$

¿From (4) in (6.1) we have

$$\text{div } \mathcal{H}^1_{ext} = 0, \quad \text{div } \left(\mathcal{H}^1_{int} + \bar{U}^1\right) + \nabla\varphi \cdot \left(U^2_z + \mathcal{H}^2_{int,z}\right) = 0,$$

$$\text{curl } \mathcal{H}^1_{int} + \nabla\varphi \wedge \mathcal{H}^2_z = 0, \quad \text{curl } \mathcal{H}^1_{ext} = 0.$$

So the equation satisfied by $\overline{\mathcal{H}^1_{int}}$ and $\mathcal{H}^1_{ext}$ reads

$$\text{div } \mathcal{H}^1_{ext} = 0, \qquad \text{curl } \mathcal{H}^1_{ext} = 0,$$

$$\text{div } \left(\overline{\mathcal{H}^1_{int}} + \overline{\bar{U}^1}\right) = 0, \qquad \text{curl } \overline{\mathcal{H}^1_{int}} = 0.$$

The transmission conditions follow from (6.2) as according to (6.13) one has $\widetilde{\mathcal{H}^1_{int}} = (\widetilde{U^1} \cdot \mathbf{n})\mathbf{n}$.

$$\mathcal{H}^1_{ext} \wedge \mathbf{n} = \overline{\mathcal{H}^1_{int}} \wedge \mathbf{n}$$
$$\mathcal{H}^1_{ext} \cdot \mathbf{n} = \left(\overline{\mathcal{H}^1_{int}} + \overline{U^1}\right) \cdot n$$

¿From these last equations one has

$$\begin{aligned} \overline{\mathcal{H}^1_{int}} &= P_{\|}(\overline{U^1})_{|\Omega} \\ \mathcal{H}^1_{ext} &= P_{\|}(\overline{U^1})_{|\Omega} \end{aligned} \tag{6.14}$$

As we will see later, we do not need the expression of $\mathbf{h}^1$.

6.2 Existence and regularity of the terms of the ansatz

We assume that that the asumptions of theorem 3.1 are satisfied. Let us recall that $T > 0$ and u^0, E^0, H^0 are given such that

$$\overline{u^0}, E^0, H^0 \in \mathcal{C}\big([0,T], p - \mathbf{H}^5(\Omega)\big)$$

solution on the limit system (3.1).
The boundary condition (6.8) needs to be satisfied exactly on the set $\{\varphi(x) = z = 0\} = \partial\Omega \times [0,\infty[$. However, we need to extend it in a convenient way to the larger set $\Omega \times [0,\infty[$, since the profile $\widetilde{U^1}(t,x,z)$ depends on x varying in Ω. and not only in $\partial\Omega$. By assumption $u^0 \in \mathcal{C}\big([0,T], \mathbf{H}^5(\Omega)\big)$, which implies that

$$\partial_{\mathbf{n}} u^0{}_{|\partial\Omega} \in \mathcal{C}\big([0,T], \mathbf{H}^{7/2}(\partial\Omega)\big).$$

Let us fix a linear continuous lifting

$$\mathcal{R} : H^{7/2}(\partial\Omega) \longrightarrow H^4(\Omega)$$

such that $\mathcal{R}u_{|\partial\Omega} = u$. The following proposition concerns the the order one profile $\widetilde{U^1}$:

Proposition 6.1 *There exists $\widetilde{U^1} : \mathbb{R}^+ \times \Omega \times \mathbb{R}^+ \longrightarrow \mathbb{R}^3$ such that :*

$$\begin{aligned}
&\partial_t \widetilde{U^1} - \widetilde{U^1}_{zz} = u^0 \wedge \widetilde{U^1}_{zz} + u^0 \wedge \widetilde{\mathcal{H}^1_{int}} + \widetilde{U^1} \wedge \left(\overline{\mathcal{H}^0} + \mathbf{h}^0\right) + \ \textit{trilinear terms} \\
&\widetilde{\mathcal{H}^1_{int}} = -\left(\widetilde{U^1}\cdot\mathbf{n}\right)\mathbf{n} \qquad\qquad (6.15)\\
&\widetilde{U^1_z} = \mathcal{R}\big(\partial_{\mathbf{n}}\overline{u^0}{}_{|\partial\Omega}\big) \quad \textit{at } z = 0
\end{aligned}$$

such that

$$\widetilde{U^1} \in \mathcal{C}^1(0,T; H^4(\Omega) \otimes H^4(\mathbb{R}^+)).$$

Moreover, if $\partial_{\mathbf{n}} u^0(0,.)\ {}_{|\partial\Omega} = 0$, the function $\widetilde{U^1}$ can be chosen (in a unique way) such that $\widetilde{U^1}_{|t=0} = 0$.

The reason why introducing the lifting $\mathcal{R}$ in the boundary conditions, and not taking the simpler condition

$$\widetilde{U^1_z}\ {}_{|z=0} = \partial_{\mathbf{n}}\overline{u^0}\,, \qquad (6.16)$$

is because of the last assertion of the proposition 6.1. The point is that, when $\partial_{\mathbf{n}} u^0(0,.)_{|\partial\Omega} = 0$, the boundary condition (6.16) is compatible with the null initial condition when the parameter x belongs to $\partial\Omega$, but *is not compatible in general when* $x \in \Omega$ since the relation $(\partial_z \widetilde{U^1}_{|z=0})_{|t=0} = (\partial_z \widetilde{U^1}_{|t=0})_{|z=0}$ does not hold in general, the term on the left being $\partial_{\mathbf{n}} u^0(0,x)$ while that on the right is 0. However, the boundary conditions of proposition 6.15, are compatible with the null initial value for the wished regularity, and for every x in Ω, since in that case $(\partial_z \widetilde{U^1}_{|z=0})_{|t=0} = \mathcal{R}(0) = 0$.

Proof: The proof is the same as for Proposition 4.2 in [10].

7 Proof of Theorem 3.1

Let $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^3, \mathbb{R})$ such that $\mathrm{supp}\psi \subset \mathcal{V}$ verifying $\psi \equiv 1$ on a neighborhood of $\partial\Omega$. Lastly, take a function $\Theta \in L^\infty(\mathbb{R}^+; H^4(\Omega))$ such that $\partial_{\mathbf{n}}\Theta(t,x) = -\partial_{\mathbf{n}}\widetilde{U^1}(t,x,0)$ on $\mathbb{R}^+ \times \partial\Omega$. We write

$(u^\varepsilon, \mathcal{H}^\varepsilon, \mathbf{h}^\varepsilon)$ as

$$
\begin{aligned}
&u^\varepsilon(t,x) = U^0(t,x) + \varepsilon\psi(x)\widetilde{U^1}\left(t, x, \frac{\varphi(x)}{\varepsilon}\right) + \varepsilon\Theta(t,x) + \varepsilon v_r^\varepsilon(t,x),\\
&\mathcal{H}^\varepsilon = P_{\|}(u^\varepsilon) = P_{\|}(U^0) + \varepsilon P_{\|}(\Theta) + \varepsilon\psi\widetilde{\mathcal{H}^1} + \varepsilon R^\varepsilon + \varepsilon P_{\|}(v_r^\varepsilon),\\
&\mathbf{h}^\varepsilon = \mathbf{h}^0 + \varepsilon \mathbf{h}_r^\varepsilon,
\end{aligned}
$$

where $R^\varepsilon = P_{\|}\left(\psi\widetilde{U^1}\right) - \psi\widetilde{\mathcal{H}^1}$.

We want to prove the following regularity for the remainder term: for all T there exists C such that :

$$
\|v_r^\varepsilon\|_{L^\infty(0,T;H^1)} + \|\varepsilon v_r^\varepsilon\|_{L^\infty(0,T;H^2)} + \|\mathbf{h}_r^\varepsilon\|_{L^\infty(0,T;H^1)} \;\leq\; C.
$$

In a first step we will write the equations satified by the remainder terms.

7.1 Equation satisfied by the remainder term

In the following, we note:

$$
\begin{aligned}
u_{app} &= U^0 + \varepsilon\psi\,\widetilde{U^1} + \varepsilon\Theta,\\
\mathcal{H}_{app} &= P_{\|}(U^0) + \varepsilon\psi\,\widetilde{\mathcal{H}^1} + \varepsilon P_{\|}(\Theta) + \varepsilon R^\varepsilon\\
\mathbf{h}_{app} &= \mathbf{h}^0
\end{aligned}
$$

Some straighforward computations show that v_r^ε solves:

$$
\begin{aligned}
&\partial_t v_r^\varepsilon - \varepsilon^2\Delta v_r^\varepsilon = T_1 + \ldots + T_{12} + F^\varepsilon \text{ on } [0,T]\times\Omega,\\
&\partial_{\mathbf{n}} v_r^\varepsilon = 0 \text{ on } \partial\Omega,\\
&v_r^\varepsilon(0,x) = 0 \text{ on } \Omega,
\end{aligned}
\tag{7.1}
$$

where

$$
\begin{aligned}
&T_1 = \varepsilon^4 v_r^\varepsilon |\nabla v_r^\varepsilon|^2,\\
&T_2 = \varepsilon^3\left(u_{app}|\nabla v_r^\varepsilon|^2 + 2v_r^\varepsilon(\nabla u_{app}, \nabla v_r^\varepsilon)\right),\\
&T_3 = \varepsilon^2\left(v_r^\varepsilon|\nabla u_{app}|^2 + 2u_{app}(\nabla v_r^\varepsilon, \nabla u_{app})\right),\\
&T_4 = \varepsilon^2 v_r^\varepsilon \wedge \Delta u_{app} + \varepsilon^2 u_{app}\wedge\Delta v_r^\varepsilon + \varepsilon^3 v_r^\varepsilon\wedge\Delta v_r^\varepsilon,\\
&T_5 = v_r^\varepsilon\wedge(\mathcal{H}_{app} + \mathbf{h}_{app}) + u_{app}\wedge P_{\|}(v_r^\varepsilon) + \varepsilon v_r^\varepsilon\wedge P_{\|}(v_r^\varepsilon),\\
&T_6 = u_{app}\wedge\mathbf{h}_r^\varepsilon + \varepsilon v_r^\varepsilon\wedge\mathbf{h}_r^\varepsilon,\\
&T_7 = -\left(u_{app}\wedge(u_{app}\wedge P_{\|}(v_r^\varepsilon)) + u_{app}\wedge(v_r^\varepsilon\wedge(\mathcal{H}_{app}+\mathbf{h}_{app}) + v_r^\varepsilon\wedge(u_{app}\wedge(\mathcal{H}_{app}+\mathbf{h}_{app})\right),\\
&T_8 = -u_{app}\wedge(u_{app}\wedge\mathbf{h}_r^\varepsilon)\\
&T_9 = -\varepsilon\left(v_r^\varepsilon\wedge(v_r^\varepsilon\wedge(\mathcal{H}_{app}+\mathbf{h}_{app})) + v_r^\varepsilon\wedge(u_{app}\wedge P_{\|}(v_r^\varepsilon)) + u_{app}\wedge(v_r^\varepsilon\wedge P_{\|}(v_r^\varepsilon))\right),\\
&T_{10} = -\varepsilon\left(v_r^\varepsilon\wedge(u_{app}\wedge\mathbf{h}_r^\varepsilon) + u_{app}\wedge(v_r^\varepsilon\wedge\mathbf{h}_r^\varepsilon)\right),\\
&T_{11} = -\varepsilon^2 v_r^\varepsilon\wedge(v_r^\varepsilon\wedge P_{\|}(v_r^\varepsilon)),\\
&T_{12} = -\varepsilon^2 v_r^\varepsilon\wedge(v_r^\varepsilon\wedge\mathbf{h}_r^\varepsilon).
\end{aligned}
$$

The term F^ε in (7.1) corresponds to

$$F^\varepsilon := -\varepsilon^{-1}\big(\partial_t u_{app} - \varepsilon^2\Delta u_{app} - \varepsilon^2 u_{app}\wedge\Delta u_{app} - |\nabla u_{app}|^2 u_{app}\big) \\ - \varepsilon^{-1}\big(- u_{app}\wedge(\mathcal{H}_{app}+\mathbf{h}_{app}) + u_{app}\wedge(u_{app}\wedge(\mathcal{H}_{app}+\mathbf{h}_{app}))\big)$$

In other way, this term reads

$$\begin{aligned} F^\varepsilon =& - \partial_t\Theta + \varepsilon A_1 + \varepsilon u_{app}|\nabla u_{app}|^2 + \varepsilon u_{app}\wedge A_1 + \varepsilon^2\Psi\widetilde{U^1}\wedge\Delta u_{app} \\ &+ u_{app}\wedge R^\varepsilon + \varepsilon\widetilde{U^1}\wedge\mathcal{H}^1 + \varepsilon\Theta\wedge\widetilde{U^1} - u_{app}\wedge(u_{app}\wedge R^\varepsilon) \\ &- \varepsilon U^0\wedge\Big((\widetilde{u^1}+\Theta)\wedge\widetilde{\mathcal{H}^1}\Big) - \varepsilon(\widetilde{u^1}+\Theta)\wedge(U^0\wedge\widetilde{\mathcal{H}^1}) \\ &- \varepsilon(\widetilde{U^1}+\Theta)\wedge\Big((\widetilde{U^1}+\Theta)\wedge P_{\|}(U^0)\Big) - \varepsilon^2(\widetilde{u^1}+\Theta)\wedge\Big((\widetilde{U^1}+\Theta)\wedge\widetilde{H^1}\Big), \end{aligned}$$

with

$$\begin{aligned} A_1 =& \Delta u_{app} - \frac{1}{\varepsilon}\Psi\widetilde{U^1_{zz}} \\ =& \Delta U^0 + \psi\Delta\varphi\widetilde{U^1_z} + 2(\nabla\psi,\nabla\varphi)\widetilde{U^1_z} + 2\psi(\nabla\varphi,\nabla\widetilde{U^1_z}) \\ &+ \varepsilon\Delta\psi\widetilde{U^1} + \varepsilon\psi\Delta\widetilde{U^1} + 2\varepsilon\nabla\psi\nabla\widetilde{U^1} + \varepsilon\Delta\Theta. \end{aligned}$$

Furthermore, we obtain that $\mathbf{h}^\varepsilon_r$ satisfies:

$$\frac{\partial^2\mathbf{h}^\varepsilon_r}{\partial t^2} - \Delta\mathbf{h}^\varepsilon_r = -\frac{\partial^2 P_\perp(v^\varepsilon_r)}{\partial t^2} \tag{7.2}$$

7.2 Estimates for the remainder terms

We will estimate the remainder term using the quantity $\mathbf{Q}$ defined by:

$$\mathbf{Q} = \|v^\varepsilon_r\|^2_{L^2} + \|\nabla v^\varepsilon_r\|^2_{L^2} + \|\varepsilon\Delta v^\varepsilon_r\|^2_{L^2} + \|\partial_t v^\varepsilon_r\|^2_{L^2} + \|\varepsilon\nabla\partial_t v^\varepsilon_r\|^2_{L^2} + \|\mathbf{h}^\varepsilon_r\|^2_{L^2} + \|\partial_t\mathbf{h}^\varepsilon_r\|^2_{L^2} + \|\nabla\mathbf{h}^\varepsilon_r\|^2_{L^2}$$

We will first obtain estimates on v^ε_r with the following

Proposition 7.1 *For all $\eta > 0$ there exists a constant $C(\eta)$ such that :*

$$\frac{1}{2}\frac{d}{dt}\left(\|v^\varepsilon_r\|^2_{L^2} + \|\nabla v^\varepsilon_r\|^2_{L^2} + \|\varepsilon\partial v^\varepsilon_r\|^2_{L^2}\right) + \varepsilon^2\left(\|\nabla v^\varepsilon_r\|^2_{L^2} + \|\varepsilon\partial v^\varepsilon_r\|^2_{L^2} + \|\varepsilon^2\nabla\Delta v^\varepsilon_r\|^2_{L^2}\right) \leq \\ \eta\|\varepsilon^2\nabla\Delta v^\varepsilon_r\|^2_{L^2} + C(\eta)\mathbf{Q} + \varepsilon C(\eta)\mathbf{Q}^9.$$

The term $\mathbf{h}^\varepsilon_r$ will satisfy a wave equation and we will obtain an estimate given by the following:

Proposition 7.2

$$\frac{d}{dt}\left(\|\partial_t\mathbf{h}^\varepsilon_r\|^2_{L^2} + \|\nabla\mathbf{h}^\varepsilon_r\|^2_{L^2}\right) \leq \|\frac{\partial^2 v^\varepsilon_r}{\partial t^2}\|_{L^2}\mathbf{Q}^{\frac{1}{2}} \tag{7.3}$$

Thus we are lead to estimate $w^\varepsilon_r := \partial_t v^\varepsilon_r$ and we have:

Proposition 7.3 *For $\eta > 0$ there exists a constant $C(\eta)$ such that :*

$$\frac{d}{dt}\left(\|w^\varepsilon_r\|^2_{L^2} + \|\varepsilon\nabla w^\varepsilon_r\|^2_{L^2}\right) + \|\varepsilon\nabla w^\varepsilon_r\|^2_{L^2} + \|\varepsilon^2\Delta w^\varepsilon_r\|^2_{L^2} \leq \eta\|\varepsilon^2\Delta w^\varepsilon_r\|^2_{L^2} + \eta\|\varepsilon^2\nabla\Delta v^\varepsilon_r\|^2_{L^2} \\ + C(\eta) + C(\eta)\mathbf{Q} + C(\eta)\varepsilon^{\frac{1}{2}}\mathbf{Q}^5$$

These technical propositions are proved in the last section.

7.3 End of the proof of Theorem 3.1

We add the inequalities obtained in lemmas 7.1, 7.2 and 7.3, and we obtain that for $\eta > 0$ there exists a constant $C(\eta)$ such that

$$\frac{1}{2}\frac{d\mathbf{Q}}{dt} + \|\nabla v_r^\varepsilon\|_{L^2}^2 + \|\varepsilon \Delta v_r^\varepsilon\|_{L^2}^2 + \|\varepsilon^2 \nabla \Delta v_r^\varepsilon\|_{L^2}^2 + \|\varepsilon \nabla w_r^\varepsilon\|_{L^2}^2 + \|\varepsilon^2 \partial w_r^\varepsilon\|_{L^2}^2 \leq$$
$$3\eta\|\varepsilon^2 \nabla \Delta v_r^\varepsilon\|_{L^2}^2 + \eta\|\varepsilon^2 \partial w_r^\varepsilon\|_{L^2}^2 + C(\eta) + C(\eta)\mathbf{Q} + C(\eta)\varepsilon^{\frac{1}{2}} P(\mathbf{Q}) + \|\frac{\partial^2 v_r^\varepsilon}{\partial t^2}\|_{L^2}\mathbf{Q}^{\frac{1}{2}} \tag{7.4}$$

Using Equation (8.4) we can estimate $\dfrac{\partial^2 v_r^\varepsilon}{\partial t^2} = \partial_t w_r^\varepsilon$ and we obtain that :

$$\begin{aligned} \|\partial_t w_r^\varepsilon\|_{L^2} &\leq \|T_1' + \ldots + T_{12}' + F_\varepsilon'\|_{L^2} + \|\varepsilon^2 \Delta w_r^\varepsilon\|_{L^2} + \varepsilon^2 \|u_{app}\|_{L^\infty}\|\Delta w_r^\varepsilon\|_{L^2} \\ &\quad + \varepsilon^3 \|v_r^\varepsilon\|_{L^\infty}\|\Delta w_r^\varepsilon\|_{L^2} \\ &\leq \eta\|\varepsilon^2 \nabla \Delta v_r^\varepsilon\|_{L^2} + \eta\|\varepsilon^2 \Delta w_r^\varepsilon\|_{L^2} + C(\eta)\left(1 + \mathbf{Q}^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}}\mathbf{Q}^{\frac{5}{2}}\right) \\ &\quad + C\|\varepsilon^2 \Delta w_r^\varepsilon\|_{L^2} + \varepsilon^{\frac{1}{2}}\mathbf{Q}^{\frac{1}{2}}\|\varepsilon^2 \Delta w_r^\varepsilon\|_{L^2} \end{aligned}$$

Hence

$$\|\frac{\partial^2 v_r^\varepsilon}{\partial t^2}\|_{L^2}\mathbf{Q}^{\frac{1}{2}} \leq C(\eta) + C(\eta)\mathbf{Q} + C(\eta)\varepsilon^{\frac{1}{2}}P(\mathbf{Q}) + 2\eta\|\varepsilon^2 \Delta w_r^\varepsilon\|_{L^2}^2 + 2\eta\|\varepsilon^2 \nabla \Delta v_r^\varepsilon\|_{L^2}^2.$$

thus using this estimate in (7.4)we obtain that :

$$\frac{1}{2}\frac{d\mathbf{Q}}{dt} + \|\nabla v_r^\varepsilon\|_{L^2}^2 + \|\varepsilon \Delta v_r^\varepsilon\|_{L^2}^2 + \|\varepsilon^2 \nabla \Delta v_r^\varepsilon\|_{L^2}^2 + \|\varepsilon \nabla w_r^\varepsilon\|_{L^2}^2 + \|\varepsilon^2 \partial w_r^\varepsilon\|_{L^2}^2 \leq$$
$$4\eta\|\varepsilon^2 \nabla \Delta v_r^\varepsilon\|_{L^2}^2 + 2\eta\|\varepsilon^2 \partial w_r^\varepsilon\|_{L^2}^2 + C(\eta) + C(\eta)\mathbf{Q} + C(\eta)\varepsilon^{\frac{1}{2}}P(\mathbf{Q})$$

We fix then $\eta > 0$ such that $4\eta < \dfrac{1}{2}$ and we obtain that there exists a constant C and a polynomial function P such that :

$$\frac{d\mathbf{Q}}{dt} + \|\nabla v_r^\varepsilon\|_{L^2}^2 + \|\varepsilon \Delta v_r^\varepsilon\|_{L^2}^2 + \|\varepsilon^2 \nabla \Delta v_r^\varepsilon\|_{L^2}^2 + \|\varepsilon \nabla w_r^\varepsilon\|_{L^2}^2 + \|\varepsilon^2 \partial w_r^\varepsilon\|_{L^2}^2 \leq C + C\mathbf{Q} + C\varepsilon^{\frac{1}{2}}P(\mathbf{Q})$$

This achieves the proof with a classical comparison argument.

8 Proof of the estimates

¿From the regularity results concerning the terms of the ansatz obtained in Section 4 we can estimate the different parameters in equation (7.1). The proof is the same as that of proposition 5.1 in paper [10].

Proposition 8.1 *For any p, $1 < p \leq +\infty$, and for any $T > 0$, there exist some constants C_p such that for any $\varepsilon > 0$ and all $t \in]0, T]$,*

$$\begin{aligned} &\|u_{app}(t,.)\|_{W^{1,p}} \leq C_p, \\ &\varepsilon\|D^2 u_{app}(t,.)\|_{L^p} \leq C_p. \end{aligned} \tag{8.1}$$

For any p, $1 < p < +\infty$, and any $T > 0$, there exist some constants C_p such that for any $\varepsilon > 0$ and all $t \in]0, T]$,

$$\varepsilon^2 \|\nabla \Delta u_{app}(t,.)\|_{L^p} \leq C_p. \tag{8.2}$$

For any p, $1 < p \leq +\infty$, and for any $T > 0$, there exist some constants C_p such that for any $\varepsilon > 0$ and all $t \in]0, T]$,

$$\begin{aligned} &\|\mathcal{H}_{app}(t,.)\|_{W^{1,p}} \leq C_p, \\ &\|F^\varepsilon(t,.)\|_{W^{1,p}} \leq C_p. \end{aligned} \tag{8.3}$$

8.1 Proof of Proposition 7.1

We recall that we denote by $\mathbf{Q}$ the following quantity:

$$\mathbf{Q} = \|v_r^\varepsilon\|_{L^2}^2 + \|\nabla v_r^\varepsilon\|_{L^2}^2 + \|\varepsilon \Delta v_r^\varepsilon\|_{L^2}^2 + \|\partial_t v_r^\varepsilon\|_{L^2}^2 + \|\varepsilon \nabla \partial_t v_r^\varepsilon\|_{L^2}^2 + \|\mathbf{h}_r^\varepsilon\|_{L^2}^2 + \|\partial_t \mathbf{h}_r^\varepsilon\|_{L^2}^2 + \|\nabla \mathbf{h}_r^\varepsilon\|_{L^2}^2$$

Lemma 8.1 *There exists a constant C such that*

$$\frac{1}{2}\frac{d}{dt}\left(\|v_r^\varepsilon\|_{L^2}^2\right) + \varepsilon^2 \|\nabla v_r^\varepsilon\|_{L^2}^2 \leq C + C\mathbf{Q} + C\varepsilon^2 \mathbf{Q}^2$$

Proof. We multiply (7.1) by v_r^ε and we obtain that

$$\frac{1}{2}\frac{d}{dt}\left(\|v_r^\varepsilon\|_{L^2}^2\right) + \varepsilon^2 \|\nabla v_r^\varepsilon\|_{L^2}^2 \leq \int_\Omega \left(T_1 + \ldots + T_{12} + F^\varepsilon\right) v_r^\varepsilon.$$

We estimate the right hand side of this inequlity on the following way:

$$\left|\int_\Omega T_1 v_r^\varepsilon\right| \leq \quad \varepsilon^4 \|v_r^\varepsilon\|_{L^\infty}^2 \|\nabla v_r^\varepsilon\|_{L^2}^2 \leq C\ \varepsilon^2 \|v_r^\varepsilon\|_{H^1}^2\ \|\varepsilon v_r^\varepsilon\|_{H^2}^2 \leq\ C\varepsilon^2\mathbf{Q}^2$$

$$\left|\int_\Omega T_2 v_r^\varepsilon\right| \leq \quad C\varepsilon^2 \|v_r^\varepsilon\|_{L^2}^{1/2}\ \|v_r^\varepsilon\|_{H^1}^{3/2}\ \|\varepsilon v^\varepsilon\|_{H^2} + C\ \varepsilon^{5/2} \|v_r^\varepsilon\|_{L^2}\ \|v_r^\varepsilon\|_{H^1}^{3/2}\ \|\varepsilon v^\varepsilon\|_{H^2}^{1/2} \leq\ C\varepsilon^2\mathbf{Q}^{3/2}$$

$$\left|\int_\Omega T_3 v_3^\varepsilon\right| \leq \quad \varepsilon^2 \|v_\varepsilon^r\|_{L^2}^2 \|\nabla u_{app}\|_{L^\infty}^2 + 2\varepsilon^2 \|u_{app}\|_{L^\infty} \|\nabla u_{app}\|_{L^\infty} \|v_\varepsilon^r\|_{L^2} \|\nabla v_\varepsilon^r\|_{L^2} \leq C\varepsilon^2\mathbf{Q}$$

$$\left|\int_\Omega T_4 v_r^\varepsilon\right| \leq \quad \varepsilon^2 \|v_\varepsilon^r\|_{L^2} \|\nabla v_\varepsilon^r\|_{L^2} \|\nabla u_{app}\|_{L^\infty} \leq C\varepsilon^2\mathbf{Q}$$

$$\left|\int_\Omega T_5 v_r^\varepsilon\right| \leq \quad \|u_{app}\|_{L^\infty} \|P_\parallel(v_\varepsilon^r)\|_{L^2} \|v_\varepsilon^r\|_{L^2} \leq C\mathbf{Q}$$

$$\left|\int_\Omega T_6 v_r^\varepsilon\right| \leq \quad \|u_{app}\|_{L^\infty} \|\mathbf{h}_\varepsilon^r\|_{L^2} \|v_\varepsilon^r\|_{L^2} \leq C\mathbf{Q}$$

$$\left|\int_\Omega T_7 v_r^\varepsilon\right| \leq \quad C\|v_\varepsilon^r\|_{L^2}^2 \leq C\mathbf{Q}$$

$$\left|\int_\Omega T_8 v_r^\varepsilon\right| \leq \quad \|u_{app}\|_{L^\infty}^2 \|\mathbf{h}_\varepsilon^r\|_{L^2} \|v_\varepsilon^r\|_{L^2} \leq C\mathbf{Q}$$

$$\left|\int_\Omega T_9 v_r^\varepsilon\right| \leq \quad C\varepsilon \|v_\varepsilon^r\|_{L^2}^{\frac{3}{2}} \|v_\varepsilon^r\|_{H^1}^{\frac{3}{2}} \leq C\varepsilon\mathbf{Q}^{\frac{3}{2}}$$

$$\left|\int_\Omega T_{10} v_r^\varepsilon\right| \leq \quad \varepsilon \|u_{app}\|_{L^\infty} \|\mathbf{h}_\varepsilon^r\|_{L^2} \|v_\varepsilon^r\|_{L^4}^2 \leq C\varepsilon\mathbf{Q}^{\frac{3}{2}}$$

$$\left|\int_\Omega T_{11} v_r^\varepsilon\right| = \quad 0$$

$$\left|\int_\Omega T_{12} v_r^\varepsilon\right| = \quad 0$$

$$\left|\int_\Omega F^\varepsilon v_r^\varepsilon\right| \leq \quad \|F^\varepsilon\|_{L^2} \|v_\varepsilon^r\|_{L^2} \leq C + \mathbf{Q}$$

Summing these estimates and remarking that $C\varepsilon\mathbf{Q}^{\frac{3}{2}} \leq C\mathbf{Q} + C\varepsilon^2\mathbf{Q}^2$ we obtain the claimed result.

Lemma 8.2 *There exists a constant C such that*

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla v_r^\varepsilon\|_{L^2}^2\right) + \varepsilon^2 \|\Delta v_r^\varepsilon\|_{L^2}^2 \leq C + C\mathbf{Q} + C\varepsilon\mathbf{Q}^2$$

Proof : we multiply (7.1) by Δv_r^ε and we obtain that

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla v_r^\varepsilon\|_{L^2}^2\right)+\varepsilon^2\|\Delta v_r^\varepsilon\|_{L^2}^2 \leq \int_\Omega \left(T_1+\ldots+T_{12}+F^\varepsilon\right)\Delta v_r^\varepsilon.$$

The terms in T_1, T_2, T_3, T_4, T_5, T_7, T_8, T_9, T_{11} and T_{12} are estimates on the following way:

$$\begin{aligned}
\left|\int_\Omega T_1\Delta v_r^\varepsilon\right| &\leq \varepsilon\|v_r^\varepsilon\|_{H^1}\|\varepsilon v_r^\varepsilon\|_{H^2}^3 \leq C\varepsilon\mathbf{Q}^2\\
\left|\int_\Omega T_2\Delta v_r^\varepsilon\right| &\leq C\sqrt{\varepsilon}\|v_r^\varepsilon\|_{H^1}^{\frac{1}{2}}\|\varepsilon v_r^\varepsilon\|_{H^2}^{\frac{5}{2}}+C\varepsilon^{\frac{3}{2}}\|v_r^\varepsilon\|_{H^1}^{\frac{3}{2}}\|\varepsilon v_r^\varepsilon\|_{H^2}^{\frac{3}{2}} \leq C\varepsilon^{\frac{1}{2}}\mathbf{Q}^{\frac{3}{2}}\\
\left|\int_\Omega T_3\Delta v_r^\varepsilon\right| &\leq C\varepsilon\|v_r^\varepsilon\|_{H^1}\|\varepsilon v_r^\varepsilon\|_{H^2} \leq C\varepsilon\mathbf{Q}\\
\left|\int_\Omega T_4\Delta v_r^\varepsilon\right| &\leq \varepsilon^2\|v_r^\varepsilon\|_{L^6}\|\nabla\Delta u_{app}\|_{L^3}\|\nabla v_r^\varepsilon\|_{L^2} \leq C\varepsilon^2\mathbf{Q}\\
\left|\int_\Omega T_5\Delta v_r^\varepsilon\right| &\leq C\|v_r^\varepsilon\|_{H^1}^2+C\varepsilon^{\frac{1}{2}}\|v_r^\varepsilon\|_{H^1}^{\frac{5}{2}}\|\varepsilon v_r^\varepsilon\|_{H^2}^{\frac{1}{2}} \leq C\mathbf{Q}+C\varepsilon^{\frac{1}{2}}\mathbf{Q}^{\frac{3}{2}}\\
\left|\int_\Omega T_7\Delta v_r^\varepsilon\right| &\leq C\|v_r^\varepsilon\|_{H^1}^2 \leq C\mathbf{Q}\\
\left|\int_\Omega T_9\Delta v_r^\varepsilon\right| &\leq C\varepsilon^{\frac{1}{2}}\|v_r^\varepsilon\|_{H^1}^{\frac{5}{2}}\|\varepsilon v_r^\varepsilon\|_{H^2}^{\frac{1}{2}}+\varepsilon\|v_r^\varepsilon\|_{H^1}^3 \leq C\varepsilon^{\frac{1}{2}}\mathbf{Q}^{\frac{3}{2}}\\
\left|\int_\Omega T_{11}\Delta v_r^\varepsilon\right| &\leq \varepsilon\|v_r^\varepsilon\|_{H^1}^3\|\varepsilon v_r^\varepsilon\|_{H^2} \leq C\varepsilon\mathbf{Q}^2\\
\left|\int_\Omega T_{12}\Delta v_r^\varepsilon\right| &\leq \varepsilon\|v_r^\varepsilon\|_{L^6}^2\|\mathbf{h}_r^\varepsilon\|_{L^6}\|\varepsilon v_r^\varepsilon\|_{L^2} \leq C\varepsilon\mathbf{Q}^2
\end{aligned}$$

For the other terms we perform an integration by parts:

$$\int_\Omega T_6 \Delta v_r^\varepsilon = -\int (\nabla u_{app} \wedge \mathbf{h}_r^\varepsilon + u_{app} \wedge \nabla \mathbf{h}_r^\varepsilon) + \varepsilon \int (u_{app} \wedge v_r^\varepsilon \wedge (\nabla \mathbf{h}_r^\varepsilon)) \nabla v_r^\varepsilon$$

$$\left| \int_\Omega T_6 \Delta v_r^\varepsilon \right| \leq \|\nabla u_{app}\|_{L^\infty} \|\mathbf{h}_r^\varepsilon\|_{L^2} \|\nabla v_r^\varepsilon\|_{L^2} + \|u_{app}\|_{L^\infty} \|\nabla \mathbf{h}_r^\varepsilon\|_{L^2} \|\nabla v_r^\varepsilon\|_{L^2}$$

$$+\varepsilon \|\nabla \mathbf{h}_r^\varepsilon\|_{L^2} \|v_r^\varepsilon\|_{L^6} \|\nabla v_r^\varepsilon\|_{L^3}$$

$$\leq C\mathbf{Q} + C\varepsilon^{\frac{1}{2}} \mathbf{Q}^{\frac{3}{2}}$$

$$\int_\Omega T_8 \Delta v_r^\varepsilon = \int (\nabla u_{app} \wedge (u_{app} \wedge \mathbf{h}_r^\varepsilon) + u_{app} \wedge (\nabla u_{app} \wedge \mathbf{h}_r^\varepsilon) + u_{app} \wedge (u_{app} \wedge \nabla \mathbf{h}_r^\varepsilon)) \nabla v_r^\varepsilon$$

$$\left| \int_\Omega T_8 \Delta v_r^\varepsilon \right| \leq C \|\mathbf{h}_r^\varepsilon\|_{H^1} \|\nabla v_r^\varepsilon\|_{L^2} \leq C\mathbf{Q}$$

$$\int_\Omega T_{10} \Delta v_r^\varepsilon = -\varepsilon \int (\nabla v_r^\varepsilon \wedge (u_{app} \wedge \mathbf{h}_r^\varepsilon) + v_r^\varepsilon \wedge (\nabla u_{app} \wedge \mathbf{h}_r^\varepsilon) + v_r^\varepsilon \wedge (u_{app} \wedge \nabla \mathbf{h}_r^\varepsilon)) \nabla v_r^\varepsilon$$

$$\left| \int_\Omega T_{10} \Delta v_r^\varepsilon \right| \leq \varepsilon \|\nabla u_{app}\|_{L^\infty} \|\mathbf{h}_r^\varepsilon\|_{L^6} \|v_r^\varepsilon\|_{L^6} \|\nabla v_r^\varepsilon\|_{L^3} + \varepsilon \|u_{app}\|_{L^\infty} \|v_r^\varepsilon\|_{L^6} \| \|\nabla \mathbf{h}_r^\varepsilon\|_{L^2} \|\nabla v_r^\varepsilon\|_{L^3}$$

$$\leq C\varepsilon^{\frac{1}{2}} \mathbf{Q}^{\frac{3}{2}}$$

$$\int_\Omega F^\varepsilon \Delta v_r^\varepsilon = -\int_\Omega \nabla F^\varepsilon \nabla v_r^\varepsilon$$

$$\left| \int_\Omega F^\varepsilon \Delta v_r^\varepsilon \right| \leq \|F^\varepsilon\|_{H^1} \|v_r^\varepsilon\|_{H^1} \leq C + \mathbf{Q}$$

We add the previous estimates and we obtain the claimed result.

Lemma 8.3 *For any fixed $\eta > 0$, there exists a constant $C(\eta)$ such that*

$$\frac{1}{2} \frac{d}{dt} \left(\|\varepsilon \Delta v_r^\varepsilon\|_{L^2}^2 \right) + \|\varepsilon^2 \nabla \Delta v_r^\varepsilon\|_{L^2}^2 \leq \eta \|\varepsilon^2 \nabla \Delta v_r^\varepsilon\|_{L^2}^2 + C(\eta)\mathbf{Q} + \varepsilon C(\eta) \mathbf{Q}^9$$

Proof : we multiply (7.1) by $\varepsilon^2 \Delta^2 v_r^\varepsilon$ and we integrate each term by part. We obtain that :

$$\frac{1}{2} \frac{d}{dt} \left(\|\varepsilon \Delta v_r^\varepsilon\|_{L^2}^2 \right) + \|\varepsilon^2 \nabla \Delta v_r^\varepsilon\|_{L^2}^2 \leq \varepsilon^2 \int_\Omega \nabla (T_1 + \ldots + T_{12} + F^\varepsilon) \nabla \Delta v_r^\varepsilon$$

with :

$$
\begin{aligned}
\left|\varepsilon^2\int_\Omega \nabla T_1\nabla\Delta v_r^\varepsilon\right| &\le \varepsilon\|\varepsilon v_r^\varepsilon\|_{H^2}^3\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}+\varepsilon\|v_r^\varepsilon\|_{H^1}^{\frac34}\|\varepsilon v_r^\varepsilon\|_{H^2}^{\frac94}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}\\
&\quad+\varepsilon\|v_r^\varepsilon\|_{H^1}^{\frac34}\|\varepsilon v_r^\varepsilon\|_{H^2}^{\frac32}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}^{\frac74}\\
&\le \frac{\eta}{13}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}^2+C(\eta)\varepsilon^2\mathbf{Q}^3+C(\eta)\varepsilon^8\mathbf{Q}^9\\
\left|\varepsilon^2\int_\Omega \nabla T_2\nabla\Delta v_r^\varepsilon\right| &\le C\varepsilon^{\frac32}\|v_r^\varepsilon\|_{H^1}^{\frac12}\|\varepsilon v_r^\varepsilon\|_{H^2}^{\frac32}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}+C\varepsilon\|\varepsilon v_r^\varepsilon\|_{H^2}^2\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}\\
&\quad+C\varepsilon^{\frac12}\|\varepsilon v_r^\varepsilon\|_{H^2}^{\frac32}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}^{\frac32}+C\varepsilon^{\frac54}\|v_r^\varepsilon\|_{H^1}^{\frac52}\|\varepsilon v_r^\varepsilon\|_{H^2}^{\frac52}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}\\
&\quad+C\varepsilon^2\|\varepsilon v_r^\varepsilon\|_{H^2}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}+C\varepsilon^{\frac32}\|\varepsilon v_r^\varepsilon\|_{H^2}^{\frac12}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}\\
&\le \frac{\eta}{13}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}^2+C(\eta)\varepsilon^2\mathbf{Q}^3+C(\eta)\\
\left|\varepsilon^2\int_\Omega \nabla T_3\nabla\Delta v_r^\varepsilon\right| &\le \varepsilon^2\|v_r^\varepsilon\|_{H^1}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}+\varepsilon\|v_r^\varepsilon\|_{L^2}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}\\
&\quad+C\varepsilon\|\varepsilon v_r^\varepsilon\|_{H^2}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}+C\varepsilon\|v_r^\varepsilon\|_{H^1}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}\\
&\le \frac{\eta}{13}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}^2+C(\eta)\varepsilon^2\mathbf{Q}\\
\left|\varepsilon^2\int_\Omega \nabla T_4\nabla\Delta v_r^\varepsilon\right| &\le C\|v_r^\varepsilon\|_{H^1}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}+C\varepsilon\|\varepsilon v_r^\varepsilon\|_{H^2}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}\\
&\quad+C\varepsilon\|\varepsilon v_r^\varepsilon\|_{H^2}^2\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}+C\varepsilon^{\frac12}\|\varepsilon v_r^\varepsilon\|_{H^2}^{\frac32}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}\\
&\le \frac{\eta}{13}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}^2+C\mathbf{Q}+C(\eta)\varepsilon^2\mathbf{Q}^3+C(\eta)\\
\left|\varepsilon^2\int_\Omega \nabla T_5\nabla\Delta v_r^\varepsilon\right| &\le C\|v_r^\varepsilon\|_{H^1}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}+C\varepsilon^{\frac12}\|v_r^\varepsilon\|_{H^1}^{\frac32}\|\varepsilon v_r^\varepsilon\|_{H^2}^{\frac12}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}\\
&\le \frac{\eta}{13}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}^2+C(\eta)\mathbf{Q}+C(\eta)\varepsilon\mathbf{Q}^2
\end{aligned}
$$

$$
\begin{aligned}
\left|\varepsilon^2\int_\Omega \nabla T_6 \nabla\Delta v_r^\varepsilon\right| &\leq \|u_{app}\|_{L^\infty}\|\nabla \mathbf{h}_r^\varepsilon\|_{L^2}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2} + \|\nabla u_{app}\|_{L^3}\|\mathbf{h}_r^\varepsilon\|_{L^6}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}\\
&\quad +\varepsilon\|\nabla v_\varepsilon^r\|_{L^3}\|\mathbf{h}_r^\varepsilon\|_{L^6}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2} + \varepsilon\|v_r^\varepsilon\|_{L^\infty}\|\nabla \mathbf{h}_r^\varepsilon\|_{L^2}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}\\
&\leq \|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}\|\nabla \mathbf{h}_r^\varepsilon\|_{L^2}\left(K + \varepsilon^{\frac{1}{2}}\|v_\varepsilon^r\|_{H^1}^{\frac{1}{2}}\|\varepsilon v_r^\varepsilon\|_{H^2}^{\frac{1}{2}}\right)\\
&\leq \frac{\eta}{13}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}^2 + C(\eta)\mathbf{Q} + C(\eta)\varepsilon\mathbf{Q}^2\\
\left|\varepsilon^2\int_\Omega \nabla T_7 \nabla\Delta v_r^\varepsilon\right| &\leq \|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}\left(\|u_{app}\|_{W^{1,\infty}}^2 + \|\mathbf{h}_{app}\|_{W^{1,\infty}}^2 + \|\mathcal{H}_{app}\|_{W^{1,\infty}}^2\right)\|v_r^\varepsilon\|_{H^1}\\
&\leq \frac{\eta}{13}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}^2 + C(\eta)\mathbf{Q}\\
\left|\varepsilon^2\int_\Omega \nabla T_8 \nabla\Delta v_r^\varepsilon\right| &\leq \|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}\left(\|u_{app}\|_{L^\infty}^2\|\nabla \mathbf{h}_r^\varepsilon\|_{L^2} + \|u_{app}\|_{W^{1,\infty}}^2\|\mathbf{h}_r^\varepsilon\|_{L^2}\right)\\
&\leq \frac{\eta}{13}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}^2 + C(\eta)\mathbf{Q}\\
\left|\varepsilon^2\int_\Omega \nabla T_9 \nabla\Delta v_r^\varepsilon\right| &\leq \|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}\left(\varepsilon\|v_r^\varepsilon\|_{L^6}\|\nabla(\mathbf{h}_{app}+\mathcal{H}_{app})\|_{L^6} + \varepsilon\|v_r^\varepsilon\|_{L^6}\|\nabla v_r^\varepsilon\|_{L^3}\|\mathbf{h}_{app}+\mathcal{H}_{app}\|_{L^\infty}\right.\\
&\quad \left.+\varepsilon\|v_r^\varepsilon\|_{L^6}^2\|\nabla u_{app}\|_{L^6} + \varepsilon\|v_r^\varepsilon\|_{L^6}\|\nabla v_r^\varepsilon\|_{L^3}\|u_{app}\|_{L^\infty}\right)\\
&\leq \frac{\eta}{13}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}^2 + C(\eta)\varepsilon\mathbf{Q}^2\\
\left|\varepsilon^2\int_\Omega \nabla T_{10} \nabla\Delta v_r^\varepsilon\right| &\leq \varepsilon\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}^2\|u_{app}\|_{W^{1,\infty}}\left(\|\nabla \mathbf{h}_r^\varepsilon\|_{L^2}\|v_r^\varepsilon\|_{L^\infty} + \|\nabla v_r^\varepsilon\|_{L^3}\|\mathbf{h}_r^\varepsilon\|_{L^6} + \|v_r^\varepsilon\|_{L^6}\|\mathbf{h}_r^\varepsilon\|_{L^2}\right)\\
&\leq \frac{\eta}{13}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}^2 + C(\eta)\varepsilon\mathbf{Q}^{\frac{5}{2}}\\
\left|\varepsilon^2\int_\Omega \nabla T_{11} \nabla\Delta v_r^\varepsilon\right| &\leq \varepsilon\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}\varepsilon^2\|\nabla v_r^\varepsilon\|_{L^6}\|v_r^\varepsilon\|_{L^6}^2\\
&\leq \frac{\eta}{13}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}^2 + C(\eta)\varepsilon^2\mathbf{Q}^3\\
\left|\varepsilon^2\int_\Omega \nabla T_{12} \nabla\Delta v_r^\varepsilon\right| &\leq \varepsilon\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}\left(\varepsilon^2\|\mathbf{h}_r^\varepsilon\|_{L^6}\|\nabla v_r^\varepsilon\|_{L^6}\|v_r^\varepsilon\|_{L^6} + \varepsilon^2\|v_r^\varepsilon\|_{L^\infty}^2\|\nabla \mathbf{h}_r^\varepsilon\|_{L^2}\right)\\
&\leq \frac{\eta}{13}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}^2 + C(\eta)\varepsilon^2\mathbf{Q}^3
\end{aligned}
$$

$$\left|\varepsilon^2\int_\Omega \nabla F^\varepsilon \nabla\Delta v_r^\varepsilon\right| \leq \|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}\|F^\varepsilon\|_{H^1}$$

$$\leq \frac{\eta}{13}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}^2 + C(\eta)$$

Adding up these estimates we conclude the proof of Proposition 8.3.

Proof of Proposition 7.1:
Adding up the estimates obtained in the Lemmas 8.1, 8.2 and 8.3 we conclude the proof of Proposition 7.1.

8.2 Proof of Proposition 7.2

We multiply (7.2) by $\mathbf{h}_r^\varepsilon$ and integrating on $\mathbb{R}^3$, and using that

$$\left\|\frac{\partial^2 P_\perp(v_r^\varepsilon)}{\partial t^2}\right\|_{L^2} = \left\|P_\perp(\frac{\partial^2 v_r^\varepsilon}{\partial t^2})\right\|_{L^2} \leq \left\|\frac{\partial^2 v_r^\varepsilon}{\partial t^2}\right\|_{L^2}$$

we conclude the proof of Proposition 7.2.

8.3 Proof of Proposition 7.3

We denote $w_r^\varepsilon = \partial_t v_r^\varepsilon$. We derivate (7.1) with respect to t and we obtain that

$$\partial_t w_r^\varepsilon - \varepsilon^2 \partial w_r^\varepsilon = T_1' + \ldots + T_{12}' + F_\varepsilon' + \varepsilon^2 u_{app}\wedge\Delta w_r^\varepsilon + \varepsilon^3 v_r^\varepsilon \wedge \Delta w_r^\varepsilon \tag{8.4}$$

where we denote $T_i' = \partial_t T_i$ for $i \neq 4$ and $T_4' = \partial_t T_4 - \left(\varepsilon^2 u_{app}\wedge\Delta w_r^\varepsilon + \varepsilon^3 v_r^\varepsilon\wedge\Delta w_r^\varepsilon\right)$ (these two last term will be treated in a special way as we will see later).

We estimate the L^2 norm of each term of the right handside in the following way :

Lemma 8.4 *For $\eta > 0$ there exists a constant $C(\eta)$ such that*

$$\|T_1' + \ldots + T_{12}' + F_\varepsilon'\|_{L^2} \leq \frac{\eta}{3}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2} + \frac{\eta}{3}\|\varepsilon^2\Delta w_r^\varepsilon\|_{L^2} + C(\eta)\left(1 + \mathbf{Q}^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}}\mathbf{Q}^{\frac{5}{2}}\right)$$

Proof : we have the following estimates :

$$
\begin{aligned}
T_1' &= \varepsilon^4 \left(w_r^\varepsilon |\nabla v_r^\varepsilon|^2 + 2 v_r^\varepsilon \nabla w_r^\varepsilon \nabla v_r^\varepsilon \right) \\
\|T_1'\|_{L^2} &\leq \varepsilon^4 \|w_r^\varepsilon\|_{L^6} \|\nabla v_r^\varepsilon\|_{L^6}^2 + 2\varepsilon^4 \|v_r^\varepsilon\|_{L^\infty} \|\nabla w_r^\varepsilon\|_{L^3} \|\nabla v_r^\varepsilon\|_{L^6} \\
&\leq \varepsilon \|\varepsilon w_r^\varepsilon\|_{H^1} \|\varepsilon v_r^\varepsilon\|_{H^2}^2 + \varepsilon \|v_r^\varepsilon\|_{H^1}^{\frac{1}{2}} \|\varepsilon v_r^\varepsilon\|_{H^2}^{\frac{3}{2}} \|\varepsilon w_r^\varepsilon\|_{H^1}^{\frac{1}{2}} \|\varepsilon^2 \partial w_r^\varepsilon\|_{L^2}^{\frac{1}{2}} + \varepsilon^{\frac{3}{2}} \|v_r^\varepsilon\|_{H^1}^{\frac{1}{2}} \|\varepsilon v_r^\varepsilon\|_{H^2}^{\frac{3}{2}} \|\varepsilon w_r^\varepsilon\|_{H^1} \\
&\leq \varepsilon \mathbf{Q}^{\frac{3}{2}} + \varepsilon \mathbf{Q}^{\frac{5}{4}} \|\varepsilon^2 \partial w_r^\varepsilon\|_{L^2}^{\frac{1}{2}} \\
T_2' &= \varepsilon^3 \left(\partial_t u_{app} |\nabla v_r^\varepsilon|^2 + 2 u_{app} \nabla v_r^\varepsilon \nabla w_r^\varepsilon + 2 w_r^\varepsilon \nabla u_{app} \nabla v_r^\varepsilon + v_r^\varepsilon \nabla \partial_t u_{app} \nabla v_r^\varepsilon + e v_r^\varepsilon \nabla u_{app} \nabla w_r^\varepsilon \right) \\
\|T_2'\|_{L^2} &\leq \varepsilon^3 \big(\|\partial_t u_{app}\|_{L^6} \|\nabla v_r^\varepsilon\|_{L^6}^2 + 2\|u_{app}\|_{L^\infty} \|\nabla v_r^\varepsilon\|_{L^6} \|\nabla w_r^\varepsilon\|_{L^3} + 2\|w_r^\varepsilon\|_{L^6} \|\nabla u_{app}\|_{L^6} \|\nabla v_r^\varepsilon\|_{L^6} \\
&\quad + \|v_r^\varepsilon\|_{L^\infty} \|\nabla \partial_t u_{app}\|_{L^2} \|\nabla v_r^\varepsilon\|_{L^\infty} + 2\|v_r^\varepsilon\|_{L^\infty} \|\nabla u_{app}\|_{L^6} \|\nabla w_r^\varepsilon\|_{L^3} \big) \\
&\leq C\varepsilon \mathbf{Q} + C\varepsilon^{\frac{1}{2}} \mathbf{Q}^{\frac{3}{4}} \|\varepsilon^2 \Delta w_r^\varepsilon\|_{L^2}^{\frac{1}{2}} + C\varepsilon \mathbf{Q}^{\frac{3}{4}} \|\varepsilon^2 \nabla \Delta v_r^\varepsilon\|_{L^2}^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
T_3' &= \varepsilon^4 \left(w_r^\varepsilon |\nabla u_{app}|^2 + 2 v_r^\varepsilon \left(\nabla \partial_t u_{app} \cdot \nabla u_{app} \right) + 2 \partial_t u_{app} \nabla v_r^\varepsilon \nabla u_{app} \right. \\
&\quad \left. + 2 u_{app} \nabla w_r^\varepsilon \nabla u_{app} + 2 u_{app} v_r^\varepsilon \nabla \partial_t u_{app} \right) \\
\|T_3'\|_{L^2} &\leq \varepsilon^4 \left(\|w_r^\varepsilon\|_{L^6} \|\nabla u_{app}\|_{L^6}^2 + 2 \|v_r^\varepsilon\|_{L^\infty} \|\nabla \partial_t u_{app}\|_{L^3} \|\nabla u_{app}\|_{L^6} + 2 \|\partial_t u_{app}\|_{L^6} \|\nabla v_r^\varepsilon\|_{L^6} \|\nabla u_{app}\|_{L^2} \right. \\
&\quad \left. + 2 \|u_{app}\|_{L^\infty} \|\nabla w_r^\varepsilon\|_{L^3} \|\nabla u_{app}\|_{L^6} + 2 \|u_{app}\|_{L^\infty} \|v_r^\varepsilon\|_{L^\infty} \|\nabla \partial_t u_{app}\|_{L^2} \right) \\
&\leq C \mathbf{Q}^{\frac{1}{2}} + C \varepsilon^{\frac{1}{2}} \mathbf{Q}^{\frac{1}{4}} \|\varepsilon^2 \Delta w_r^\varepsilon\|_{L^2}^{\frac{1}{2}} \\
T_4' &= \varepsilon^2 \left(w_r^\varepsilon \wedge \Delta u_{app} + v_r^\varepsilon \wedge \Delta \partial_t u_{app} + \partial_t u_{app} \wedge \Delta v_r^\varepsilon + \varepsilon w_r^\varepsilon \wedge \Delta v_r^\varepsilon \right) \\
\|T_4'\|_{L^2} &\leq \varepsilon^2 \left(\|w_r^\varepsilon\|_{L^\infty} \|\Delta u_{app}\|_{L^2} + \|v_r^\varepsilon\|_{L^\infty} \|\Delta \partial_t u_{app}\|_{L^2} + \|\partial_t u_{app}\|_{L^\infty} \|\Delta v_r^\varepsilon\|_{L^2} + \varepsilon \|w_r^\varepsilon\|_{L^\infty} \|\Delta v_r^\varepsilon\|_{L^2} \right) \\
&\leq C \varepsilon \mathbf{Q}^{\frac{1}{2}} + C \varepsilon^{\frac{1}{2}} \mathbf{Q}^{\frac{3}{4}} \|\varepsilon^2 \Delta w_r^\varepsilon\|_{L^2}^{\frac{1}{2}} + C \varepsilon^{\frac{1}{2}} \mathbf{Q} \\
T_5' &= w_r^\varepsilon \wedge \mathcal{H}_{app} + v_r^\varepsilon \wedge \partial_t \mathcal{H}_{app} + \partial_t u_{app} \wedge P_\parallel(v_r^\varepsilon) + u_{app} \wedge \partial_t P_\parallel(v_r^\varepsilon) + \varepsilon w_r^\varepsilon \wedge P_\parallel(v_r^\varepsilon) + \varepsilon v_r^\varepsilon \wedge \partial_t P_\parallel(v_r^\varepsilon) \\
\|T_5'\|_{L^2} &\leq \|w_r^\varepsilon\|_{L^2} \|\mathcal{H}_{app}\|_{L^\infty} + \|v_r^\varepsilon\|_{L^6} \|\partial_t \mathcal{H}_{app}\|_{L^3} + \|\partial_t u_{app}\|_{L^\infty} \|v_r^\varepsilon\|_{L^2} + \|u_{app}\|_{L^\infty} \|w_r^\varepsilon\|_{L^2} \\
&\quad + \varepsilon \|w_r^\varepsilon\|_{L^3} \|v_r^\varepsilon\|_{L^6} + \varepsilon \|v_r^\varepsilon\|_{L^6} \|w_r^\varepsilon\|_{L^3} \\
&\leq C \mathbf{Q}^{\frac{1}{2}} + C \varepsilon^{\frac{1}{2}} \mathbf{Q} \\
T_6' &= \partial_t u_{app} \wedge \mathbf{h}_r^\varepsilon + u_{app} \wedge \partial_t \mathbf{h}_r^\varepsilon + \varepsilon w_r^\varepsilon \wedge \mathbf{h}_r^\varepsilon + \varepsilon v_r^\varepsilon \wedge \partial_t \mathbf{h}_r^\varepsilon \\
\|T_6'\|_{L^2} &\leq \|\partial_t u_{app}\|_{L^\infty} \|\mathbf{h}_r^\varepsilon\|_{L^2} + \|u_{app}\|_{L^\infty} \|\partial_t \mathbf{h}_r^\varepsilon\|_{L^2} + \varepsilon \|w_r^\varepsilon\|_{L^3} \|\mathbf{h}_r^\varepsilon\|_{L^6} + \varepsilon \|v_r^\varepsilon\|_{L^\infty} \|\partial_t \mathbf{h}_r^\varepsilon\|_{L^2} \\
&\leq C \mathbf{Q}^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}} \mathbf{Q} \\
T_7' &= \partial_t u_{app} \wedge (u_{app} \wedge P_\parallel(v_r^\varepsilon)) + u_{app} \wedge (\partial_t u_{app} \wedge P_\parallel(v_r^\varepsilon)) + u_{app} \wedge (u_{app} \wedge \partial_t P_\parallel(v_r^\varepsilon)) \\
&\quad + \partial_t u_{app} \wedge (v_r^\varepsilon \wedge (\mathbf{h}_{app} + \mathcal{H}_{app})) + u_{app} \wedge (w_r^\varepsilon \wedge (\mathbf{h}_{app} + \mathcal{H}_{app})) \\
&\quad + u_{app} \wedge (v_r^\varepsilon \wedge \partial_t (\mathbf{h}_{app} + \mathcal{H}_{app}))
\end{aligned}
$$

$$
\begin{aligned}
\|T_7'\|_{L^2} &\leq \|\partial_t u_{app}\|_{L^\infty}\|u_{app}\|_{L^\infty}\|v_r^\varepsilon\|_{L^2} + \|u_{app}\|_{L^\infty}\|\partial_t u_{app}\|_{L^\infty}\|v_r^\varepsilon\|_{L^2} + \|u_{app}\|_{L^\infty}\|u_{app}\|_{L^\infty}\|w_r^\varepsilon\|_{L^2} \\
&\quad +\|\partial_t u_{app}\|_{L^\infty}\|v_r^\varepsilon\|_{L^2}\|\mathbf{h}_{app} + \mathcal{H}_{app}\|_{L^\infty} + \|u_{app}\|_{L^\infty}\|w_r^\varepsilon\|_{L^2}\|\mathbf{h}_{app} + \mathcal{H}_{app}\|_{L^\infty} \\
&\quad +\|u_{app}\|_{L^\infty}\|v_r^\varepsilon\|_{L^2}\|\partial_t(\mathbf{h}_{app} + \mathcal{H}_{app})\|_{L^\infty} \\
&\leq C\mathbf{Q}^{\frac{1}{2}} \\
T_8' &= \partial_t u_{app} \wedge (u_{app} \wedge \mathbf{h}_r^\varepsilon) + u_{app} \wedge (\partial_t u_{app} \wedge \mathbf{h}_r^\varepsilon) + u_{app} \wedge (u_{app} \wedge \partial_t h_r^\varepsilon) \\
\|T_8'\|_{L^2} &\leq 2\|\partial_t u_{app}\|_{L^\infty}\|u_{app}\|_{L^\infty}\|\mathbf{h}_r^\varepsilon\|_{L^2} + \|u_{app}\|_{L^\infty}^2\|\partial_t h_r^\varepsilon\|_{L^2} \\
&\leq C\mathbf{Q}^{\frac{1}{2}} \\
T_9' &= \varepsilon\,(w_r^\varepsilon \wedge (v_r^\varepsilon \wedge (\mathbf{h}_{app} + \mathcal{H}_{app})) + v_r^\varepsilon \wedge (w_r^\varepsilon \wedge (\mathbf{h}_{app} + \mathcal{H}_{app})) + v_r^\varepsilon \wedge (v_r^\varepsilon \wedge \partial_t(\mathbf{h}_{app} + \mathcal{H}_{app})) \\
&\quad +w_r^\varepsilon \wedge (u_{app} \wedge P_{\|}(v_r^\varepsilon)) + v_r^\varepsilon \wedge (\partial_t u_{app} \wedge P_{\|}(v_r^\varepsilon)) + v_r^\varepsilon \wedge (u_{app} \wedge \partial_t P_{\|}(v_r^\varepsilon)) \\
&\quad +\partial_t u_{app} \wedge (v_r^\varepsilon \wedge P_{\|}(v_r^\varepsilon)) + u_{app} \wedge w_r^\varepsilon \wedge P_{\|}(v_r^\varepsilon)) + u_{app} \wedge (v_r^\varepsilon \wedge \partial_t(P_{\|}(v_r^\varepsilon)))\big) \\
\|T_9'\|_{L^2} &\leq \varepsilon\,\big(2\varepsilon\|w_r^\varepsilon\|_{L^3}\|v_r^\varepsilon\|_{L^6}\|\mathbf{h}_{app} + \mathcal{H}_{app}\|_{L^\infty} + \|v_r^\varepsilon\|_{L^6}^2\|\partial_t(\mathbf{h}_{app} + \mathcal{H}_{app})\|_{L^6} \\
&\quad + 2\|w_r^\varepsilon\|_{L^3}\|u_{app}\|_{L^\infty}\|v_r^\varepsilon\|_{L^6} + 2\|v_r^\varepsilon\|_{L^3}\|\partial_t u_{app}\|_{L^\infty}\|v_r^\varepsilon\|_{L^6} + 2\|v_r^\varepsilon\|_{L^6}\|u_{app}\|_{L^\infty}\|w_r^\varepsilon\|_{L^3}) \\
&\quad \leq C\varepsilon^{\frac{1}{2}}\mathbf{Q} \\
T_{10}' &= \varepsilon\,(w_r^\varepsilon \wedge (u_{app} \wedge \mathbf{h}_r^\varepsilon) + v_r^\varepsilon \wedge (\partial_t u_{app} \wedge \mathbf{h}_r^\varepsilon) + v_r^\varepsilon \wedge (u_{app} \wedge \partial_t \mathbf{h}_r^\varepsilon) \\
&\quad + \partial_t u_{app} \wedge (v_r^\varepsilon \wedge \mathbf{h}_r^\varepsilon) + u_{app} \wedge (w_r^\varepsilon \wedge \mathbf{h}_r^\varepsilon) + u_{app} \wedge (v_r^\varepsilon \wedge \partial_t \mathbf{h}_r^\varepsilon)) \\
\|T_{10}'\|_{L^2} &\leq 2\varepsilon\,(\|w_r^\varepsilon\|_{L^3}\|u_{app}\|_{L^\infty}\|\mathbf{h}_r^\varepsilon\|_{L^6} + \|v_r^\varepsilon\|_{L^3}\|\partial_t u_{app}\|_{L^\infty}\|\mathbf{h}_r^\varepsilon\|_{L^6} + \|v_r^\varepsilon\|_{L^\infty}\|u_{app}\|_{L^\infty}\|\partial_t \mathbf{h}_r^\varepsilon\|_{L^2}) \\
&\leq C\varepsilon^{\frac{1}{2}}\mathbf{Q} \\
T_{11}' &= \varepsilon^2\,\big(w_r^\varepsilon \wedge (v_r^\varepsilon \wedge P_{\|}(v_r^\varepsilon)) + v_r^\varepsilon \wedge (w_r^\varepsilon \wedge P_{\|}(v_r^\varepsilon)) + v_r^\varepsilon \wedge (v_r^\varepsilon \wedge \partial_t P_{\|}(v_r^\varepsilon))\big) \\
\|T_{11}'\|_{L^2} &\leq 3\varepsilon^2\|w_r^\varepsilon\|_{L^6}\|v_r^\varepsilon\|_{L^6}^2 \\
&\leq C\varepsilon\mathbf{Q}^{\frac{3}{2}} \\
T_{12}' &= \varepsilon^2\,(w_r^\varepsilon \wedge (v_r^\varepsilon \wedge \mathbf{h}_r^\varepsilon) + v_r^\varepsilon \wedge (w_r^\varepsilon \wedge \mathbf{h}_r^\varepsilon) + v_r^\varepsilon \wedge (v_r^\varepsilon \wedge \partial_t \mathbf{h}_r^\varepsilon)) \\
\|T_{12}'\|_{L^2} &\leq \varepsilon^2\,(2\|w_r^\varepsilon\|_{L^6}\|v_r^\varepsilon\|_{L^6}\|\mathbf{h}\|_{L^2} + \|v_r^\varepsilon\|_{L^\infty}\|\partial_t \mathbf{h}_r^\varepsilon\|_{L^2}) \\
&\leq C\varepsilon\mathbf{Q}^{\frac{3}{2}}
\end{aligned}
$$

In addition we know that $\|\partial_t F^\varepsilon\|_{L^2} \leq C$.

Adding up these estimates, and using Young inequality for the terms $\|\varepsilon^2\Delta w_r^\varepsilon\|_{L^2}$ and $\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}$, we conclude the proof of Lemma 8.4.

Proof of Proposition 7.3

Multiplying (8.4) by w_r^ε, and using Proposition 8.4 we obtain that

$$\begin{aligned}
\frac{d}{dt}\|w_r^\varepsilon\|_{L^2}^2 + \|\varepsilon\nabla w_r^\varepsilon\|_{L^2}^2 &\leq \left(\tfrac{\eta}{3}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2} + \tfrac{\eta}{3}\|\varepsilon^2\Delta w_r^\varepsilon\|_{L^2} + C(\eta)\left(1+\mathbf{Q}^{\frac12}+\varepsilon^{\frac12}\mathbf{Q}^{\frac52}\right)\right)\|w_r^\varepsilon\|_{L^2} \\
&\quad +\varepsilon^2\|u_{app}\|_{L^\infty}\|\Delta w_r^\varepsilon\|_{L^2}\|w_r^\varepsilon\|_{L^2} + \varepsilon^3\|v_r^\varepsilon\|_{L^6}\|w_r^\varepsilon\|_{L^3}\|\Delta w_r^\varepsilon\|_{L^2} \\
&\leq C(\eta) + C(\eta)\mathbf{Q} + 2\eta\|\varepsilon^2\partial w_r^\varepsilon\|_{L^2}^2 + 2\eta\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}^2 + \varepsilon^{\frac12}\mathbf{Q}^3
\end{aligned}$$

We multiply then (8.4) by $\varepsilon^2\Delta w_r^\varepsilon$, we remark that :

$$\int_\Omega \left(\varepsilon^2 u_{app}\wedge\Delta w_r^\varepsilon + \varepsilon^3 v_r^\varepsilon\wedge\Delta w_r^\varepsilon\right)\Delta w_r^\varepsilon = 0.$$

Thus we obtain that :

$$\begin{aligned}
\frac{d}{dt}\|\varepsilon\nabla w_r^\varepsilon\|_{L^2}^2 + \|\varepsilon^2\Delta w_r^\varepsilon\|_{L^2}^2 &\leq \frac{\eta}{3}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}\|\varepsilon^2\Delta w_r^\varepsilon\|_{L^2} + \frac{\eta}{3}\|\varepsilon^2\Delta w_r^\varepsilon\|_{L^2}^2 \\
&\quad +C(\eta)\left(1+\mathbf{Q}^{\frac12}+\varepsilon^{\frac12}\mathbf{Q}^{\frac52}\right)\|\varepsilon^2\Delta w_r^\varepsilon\|_{L^2} \\
&\leq \frac{\eta}{3}\|\varepsilon^2\Delta w_r^\varepsilon\|_{L^2}^2 + \frac{\eta}{3}\|\varepsilon^2\nabla\Delta v_r^\varepsilon\|_{L^2}^2 + C(\eta)\mathbf{Q} + C(\eta)\varepsilon^{\frac12}\mathbf{Q}^5
\end{aligned}$$

We add up these two inequalities and that concludes the proof of Proposition 7.3.

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