Stability of Static Walls for a three dimensional Model of Ferromagnetic Material

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Projet ANR SICOMAF

Abstract. In this paper we consider a three dimensional model of ferromagnetic material. We deal with the static domain wall configuration calculated by Walker. We prove the stability of this configuration for the Landau-Lifschitz equation with a simplified expression of the demagnetizing field.

Résumé. Dans cet article, on considère un modèle tridimensionnel de matériau ferromagnétique. On étudie les profils de murs statiques calculés initialement par Walker. On démontre la stabilité de ces profils pour l'équation de Landau-Lifschitz avec un modèle simplifié pour le champ démagnétisant.

MSC: 35K55, 35Q60.

Keywords: Landau-Lifschitz equation, domain walls, stability.

1 Introduction and main results

The formation and the dynamics of domain walls are among the most studied topics in micromagnetism. In his pioneering works [29], Walker performed the exact integration of the equations of motion for a planar wall (see [26]). In this paper, we tackle the problem of the stability of these exact solutions for the Landau-Lifschitz equation in a simplified 3-dimensional model.
Let us recall the general framework of the ferromagnetism (see [5], [17] and [27]). We consider an infinite homogeneous ferromagnetic medium. We denote by m the magnetization:

$$\begin{array}{rccc} m: & \mathbb{R}^+ \times \mathbb{R}^3 & \to & \mathbb{R}^3 \\ & (t,x,y,z) & \mapsto & m(t,x,y,z). \end{array}$$

The magnetic moment m links the magnetic induction B and the magnetic field H by the relation $B = m + H$. In addition, we assume that the material is saturated so that the magnitude of m is constant. After renormalization we assume that

$$|m| = 1 \text{ at any point.} \tag{1.1}$$

The evolution of m is described by the Landau-Lifschitz equation:

$$\partial_t m = -m \times H_{eff} - m \times (m \times H_{eff}). \tag{1.2}$$

The effective field $H_{eff} = -\nabla \mathcal{E}$ is derived from the micromagnetism energy $\mathcal{E}$ given by

$$\mathcal{E} = \mathcal{E}_{exch} + \mathcal{E}_{dem} + \mathcal{E}_{anis},$$

where

- the exchange energy $\mathcal{E}_{exch}$ writes

$$\mathcal{E}_{exch} = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla m|^2,$$

- the anisotropy energy reflects the existence of a preferential axis of magnetization:

$$\mathcal{E}_{anis} = \frac{1}{2}\int_{\mathbb{R}^3}(1-|m_3|^2), \quad m=(m_1,m_2,m_3).$$

- $\mathcal{E}_{dem}$ is the demagnetizing energy:

$$\mathcal{E}_{dem} = \frac{1}{2}\int_{\mathbb{R}^3}|h_d(m)|^2.$$

The demagnetizing field $h_d(m)$ is characterized by

$$\begin{cases} \text{curl } h_d(m) = 0, \\ \text{div } (h_d(m)+m) = 0. \end{cases} \tag{1.3}$$

Therefore we obtain that

$$H_{eff} = \Delta m + m_3 e_3 + h_d(m),$$

where e_3 is the third vector of the canonical basis (e_1, e_2, e_3) of $\mathbb{R}^3$.

Existence results for the Landau-Lifschitz equation can be found in [2], [6], [14], [16], [20] and [28] for the weak solutions, and in [7], [8] and [9] for the strong solutions. Numerical simulations are performed in [3], [4], [21], [22] and [23].

In case of a magnetic moment only depending on the x variable, the demagnetizing field obtained by integrating (1.3) reads $h_d(m) = -m_1 e_1$. With this expression of the demagnetizing field, Walker calculated in [26] the following static solution to the Landau-Lifschitz equation:

$$M_0(x,y,z) = M_0(x) = \begin{pmatrix} 0 \\ 1/\mathrm{ch}\, x \\ -\mathrm{th}\, x \end{pmatrix}. \tag{1.4}$$

The profile M_0 modelizes a domain wall connecting the domain $\{x \to -\infty\}$ in which $m \sim e_3$ with the domain $\{x \to +\infty\}$ in which $m \sim -e_3$.

In our paper we simplify the model assimilating h_d to $-m_1 e_1$ even for perturbations of M_0. So we deal with the following system:

$$\begin{aligned} &\partial_t m = -m \times H_{eff} - m \times (m \times H_{eff}), \\ &H_{eff} = \Delta m + m_3 e_3 - m_1 e_1, \end{aligned} \tag{1.5}$$

and we adress the stability of the static solution M_0 for the system (1.5). Our main result is the following:

Theorem 1.1. *Let $\varepsilon > 0$. There exists $\delta > 0$ such that for all $m_0 \in H^2(\mathbb{R}^3;\mathbb{R}^3)$, if m_0 satisfies the saturation constraint $|m_0| = 1$ and verifies $\|m_0 - M_0\|_{H^2(\mathbb{R}^3)} \leq \delta$, then the solution m of the Landau-Lifschitz equation (1.5) together with the initial data $m(0,x,y,z) = m_0(x,y,z)$ satisfies*

$$\forall\, t \geq 0, \|m(t,). - M_0\|_{H^2(\mathbb{R}^3)} \leq \varepsilon.$$

In [10], we proved the same kind of stability result for a one dimensional model of ferromagnetic nanowire. We extended this result in [11] by proving the controllability of the wall position for this 1-d model. In the present paper, we deal with the 3-d model (1.5). The proof of the stability result somewhat follows that presented in [10]. The first two steps are formally similar.

At the begining we must consider perturbations m of the profile M_0 satisfying the physical constraint $|m| = 1$. In order to do that, we describe m in the mobile frame $(M_0(x), M_1(x), M_2)$ where

$$M_1(x) = \begin{pmatrix} 0 \\ \operatorname{th} x \\ \dfrac{1}{\operatorname{ch} x} \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

writing

$$m(t,x,y,z) = r_1(t,x,y,z)M_1(x) + r_2(t,x,y,z)M_2 + \left(1 - (r_1(t,x,y,z))^2 - (r_2(t,x,y,z))^2\right)^{\frac{1}{2}} M_0(x).$$

The new unknown $r = (r_1, r_2)$ takes its values in the flat space $I\!R^2$. Then we rewrite the Landau-Lifschitz equation with the unknown r, and we obtain in Section 2 that the Landau-Lifschitz equation is equivalent to a nonlinear equation on r, and the stability of M_0 is equivalent to the stability of 0 for this new equation.

Now the problem is that the linearized of the new equation around zero admits 0 as a simple eigenvalue. This is due to the invariance of the Landau-Lifschitz equation (1.5) by translation in the x-variable (see Section 3). Following the method developped in [30], [15], [18] and [19] (for travelling waves solutions to semilinear parabolic equations), we decompose the perturbations into a spacial translation component (the "front") and a normal component. The front satisfies a quasilinear parabolic equation the linearized of which behaves like the heat flow in $I\!R^2$. The normal component is shown to satisfy a very dissipative quasilinear parabolic equation (see Section 4).

Section 5 is devoted to variational estimates to prove the stability. The situation in the present paper is much more complicated than the one dimensional case, because in 1-d, the front part satisfies an ordinary differential equation. In addition, here the equations are quasilinear, and Kapitula's method with semigroup estimates for the heat flow cannot be applied (see [18] for example).

Our method is the one used to prove a global existence with small data result. In the variational estimates, the good sign terms induced by the linear part enable us to absorb the nonlinear terms. In our case, the L^2 norm of the front does not appear as an absorbing term. It's the same thing for the heat flow in the whole space. This dissipation defect for the front is compensated by a careful study of the nonlinear part. The key point is that we can control this nonlinear part by the gradient of the front (see Section 6).

Remark 1.1. *When a constant magnetic field is applied in the x-direction on the ferromagnetic material, it is observed that the domain wall is translated in the x-direction. In [26] such solutions are calculated. They are described as travelling waves of a profile obtained from M_0 by rotation and dilation. The stability of these moving walls remains an open problem and our method does not work in that case. In the same way, the stability of walls with the non simplified demagnetizing field remains unproved (see Remark 4.1 below).*

Remark 1.2. *In the static case, the formation of domain walls is explained by asymptotic methods. We refer the interested reader to [1], [12], [13] and [25].*

2 Mobile frame

We consider the mobile frame $(M_0(x), M_1(x), M_2)$ given by:

$$\forall x \in I\!R, \quad M_0(x) = \begin{pmatrix} 0 \\ 1/\operatorname{ch} x \\ -\operatorname{th} x \end{pmatrix}, \; M_1(x) = \begin{pmatrix} 0 \\ \operatorname{th} x \\ 1/\operatorname{ch} x \end{pmatrix}, \; M_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Let us introduce the smooth map $\nu : B(0,1) \to I\!R$ defined for $\xi = (\xi_1, \xi_2)$ by

$$\nu(\xi) = \sqrt{1 - (\xi_1)^2 - (\xi_2)^2} - 1,$$

where $B(0,1) = \{(\xi_1, \xi_2),\ \ (\xi_1)^2 + (\xi_2)^2 < 1\}$ is the unit ball of $\mathbb{R}^2$.
We write the perturbations of M_0 as:

$$m(t,x,y,z) = M_0(x) + r_1(t,x,y,z)M_1(x) + r_2(t,x,y,z)M_2(x) + \nu(r(t,x,y,z))M_0(x),$$

so that the constraint $|m| = 1$ is satisfied.
We will work with the unknown $r(t,x,y,z) = \begin{pmatrix} r_1(t,x,y,z) \\ r_2(t,x,y,z) \end{pmatrix}$.

We remark that we have $r_1(t,x,y,z) = m(t,x,y,z)\cdot M_1(x)$ and $r_2(t,x,y,z) = m(t,x,y,z)\cdot M_2$.

After a rather long algebraic calculation, we obtain that if m satisfies (1.5) then r verifies:

$$\partial_t r = \Lambda r + F(x, r, \nabla r, \Delta r), \tag{2.6}$$

where

$$\Lambda r = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} Lr_1 \\ Lr_2 + r_2 \end{pmatrix},$$

with $L = -\Delta + f,\quad f(x) = 2\mathrm{th}^2 x - 1$.
The nonlinear part $F : \mathbb{R}\times B(0,1)\times \mathbb{R}^4 \times \mathbb{R}^2 \to \mathbb{R}^2$ is defined by:

$$F(x, r, \nabla r, \Delta r) = A(r)\Delta r + \sum_{i=1}^{3} B(r)(\partial_i r, \partial_i r) + C(x,r)(\partial_x r) + D(x,r),$$

with the following notations:

- $A \in \mathcal{C}^\infty(B(0,1); \mathcal{M}_2(\mathbb{R}))$ ($\mathcal{M}_2(\mathbb{R})$ is the set of the real 2×2 matrices):

$$A(r) = \begin{pmatrix} -(r_1)^2 & \nu(r) - r_1 r_2 \\ \nu(r) - r_1 r_2 & -(r_2)^2 \end{pmatrix} + \begin{pmatrix} -r_2 - (1+\nu(r))r_1 \\ r_1 - (1+\nu(r))r_2 \end{pmatrix} \nu'(r),$$

- $B \in \mathcal{C}^\infty(B(0,1); \mathcal{L}_2(\mathbb{R}^2))$ ($\mathcal{L}_2(\mathbb{R}^2;\mathbb{R}^2)$ is the set of the bilinear functions defined on $\mathbb{R}^2\times\mathbb{R}^2$ with values in $\mathbb{R}^2$):

$$B(r)(\xi,\xi) = \begin{pmatrix} -r_2 - r_1 - r_1\nu(r) \\ r_1 - r_2 - r_2\nu(r) \end{pmatrix} \nu''(r)(\xi,\xi),$$

- $\partial_1 r = \partial_x r = \dfrac{\partial r}{\partial x}$, $\partial_2 r = \dfrac{\partial r}{\partial y}$, $\partial_3 r = \dfrac{\partial r}{\partial z}$,
- $C \in \mathcal{C}^\infty(\mathbb{R}\times B(0,1); \mathcal{M}_2(\mathbb{R}))$:

$$C(x,r)(\xi) = \frac{2}{\mathrm{ch}\, x} \begin{pmatrix} -r_2 - r_1 - r_1\nu(r) \\ r_1 - r_2 - r_2\nu(r) \end{pmatrix} \xi_1 + \frac{2}{\mathrm{ch}\, x} \begin{pmatrix} -1 + (r_1)^2 \\ 1 + \nu(r) + r_1 r_2 \end{pmatrix} \nu'(r)(\xi),$$

- $D \in \mathcal{C}^\infty(\mathbb{R}\times B(0,1); \mathbb{R}^2)$: $D(x,r) = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}$ with

$$D_1 = -\Big(r_2 + r_2 f + 2fr_1 + \ fr_1\nu(r)\Big)\,\nu(r) + (r_2)^2 r_1 + \frac{2\mathrm{sh}\, x}{\mathrm{ch}^2 x} r_1\left(r_2 + r_1 + r_1\nu(r)\right),$$

$$D_2 = \Big(fr_1 - 2fr_2 - fr_2\nu(r)\ - 2r_2 - r_2\nu(r)\Big)\,\nu(r) - r_2(r_1)^2 - \frac{2\mathrm{sh}\, x}{\mathrm{ch}^2 x} r_1\left(r_1 - r_2 - r_2\nu(r)\right).$$

In fact, both forms of the Landau-Lifschitz equation are equivalent as it is stated in the following proposition:

Proposition 2.1. *Let $m \in \mathcal{C}^1(0,T; H^2(I\!\!R^3; I\!\!R^3))$ such that $|m| = 1$ and satisfying*

$$\forall\, t \in [0,T[,\ \forall\,(x,y,z) \in I\!\!R^3,\ |m(t,x,y,z) - M_0(x)| < \sqrt{2}. \tag{2.7}$$

We introduce $r = (r_1, r_2) \in \mathcal{C}^1(0,T; H^2(I\!\!R^2; I\!\!R^2))$ defined by

$$m(t,x,y,z) = M_0(x) + r_1(t,x,y,z)M_1(x) + r_2(t,x,y,z)M_2(x) + \nu(r(t,x,y,z))M_0(x)$$

(Assumption (2.7) implies that $r(t,x,y,z) \in B(0,1)$ for all (t,x,y,z)).
Then m is solution to the Landau-Lifschitz equation (1.5) if and only if r is solution to (2.6) and M_0 is stable for (1.5) if and only if 0 is stable for (2.6).

Sketch of the proof. By projection on M_1 and M_2, it is clear that if m satisfies the Landau-Lifschitz equation (1.5) then m satisfies (2.6). The converse is proved in [10] using the fact that if $|m| = 1$ and if m satisfies the projection of (1.5) onto $I\!\!R M_1$ and $I\!\!R M_2$, then it satisfies (1.5).

Let us estimate the nonlinear functions appearing in (2.6). Since $\nu(\xi) = \mathcal{O}(|\xi|^2)$, by straightforward calculations, we obtain the following proposition:

Proposition 2.2. *There exists a constant K such that for $r \in B(0,1/2)$ and for $x \in I\!\!R$,*

- $|A(r)| \leq K|r|^2$ *and* $|A'(r)| \leq K|r|$,
- $|B(r)| \leq K|r|$ *and* $|B'(r)| \leq K$,
- $|C(x,r)| \leq \dfrac{K}{ch\,x}|r|$ *and* $|\partial_r C(x,r)| \leq \dfrac{K}{ch\,x}$,
- $|D(x,r)| \leq K|r|^3 + \dfrac{K}{ch\,x}|r|^2$ *and* $|\partial_r D(x,r)| \leq K|r|^2 + \dfrac{K}{ch\,x}|r|$.

3 Linear properties

We denote by L the linear operator acting on $H^2(I\!\!R^3)$ defined by

$$Lu = -\Delta u + fu,$$

with $f(x,y,z) = 2\mathrm{th}^2 x - 1$.
We denote by L_1 the reduced operator acting on $H^2(I\!\!R)$ given by

$$L_1 = -\partial_{xx} + f.$$

Proposition 3.1. *The operator L_1 is positive symmetric. Its spectrum is $\{0\} \cup [1,+\infty[$, where 0 is the unique eigenvalue, and $[1,+\infty[$ is the essential spectrum. In addition, 0 is simple.*

Proof. On one hand, since $f(x) = 2\mathrm{th}^2 x - 1$, the essential spectrum is $[1,+\infty[$ (see the Weyl Theorem in [24]).
On the other hand, $L_1 = l^* \circ l$ where $l = \partial_x + \mathrm{th}\, x$. So L_1 is positive. The kernel of L_1 is directed by $\dfrac{1}{\mathrm{ch}\, x}$:

$$\mathrm{Ker}\ L_1 = \mathrm{Ker}\ l = I\!\!R \frac{1}{\mathrm{ch}\, x}.$$

Finally we have $l \circ l^* = -\partial_{xx} + 1$, so if v is an eigenvector associated to the eigenvalue λ, then

$$l \circ l^* \circ lv = \lambda lv,$$

that is, if $v \notin \mathrm{Ker}\ l$, then λ is an eigenvector for $-\partial_{xx} + 1$, which leads to a contradiction.

Remark 3.1. *As we remarked in [10] and [11], a direct consequence of Proposition 3.1 is the following. Let $\mathcal{E}_1$ defined by*

$$\mathcal{E}_1 = (Ker\, L_1)^\perp = \left\{ v \in H^2(I\!R), \int_{I\!R} v(x) \frac{1}{ch\, x} dx = 0 \right\}.$$

Then on $\mathcal{E}_1$, the H^2-norm is equivalent to $\|L_1 u\|_{L^2(I\!R)}$ and the H^3-norm is equivalent to $\|L_1^{\frac{3}{2}} u\|_{L^2(I\!R)}$.

Proposition 3.2. *The operator $L = -\Delta + f$ is a positive self-adjoint operator defined on $H^2(I\!R^3)$. Let us consider $\mathcal{E}$ defined by*

$$\mathcal{E} = \left\{ v \in H^2(I\!R^3), \forall\, (y, z) \in I\!R^2, \int_{x \in I\!R} v(x, y, z) \frac{1}{ch\, x} dx = 0 \right\}.$$

There exists K such that

$$\forall\, v \in \mathcal{E}, \|v\|_{H^2(I\!R^3)} \leq K \|Lv\|_{L^2(I\!R^3)},$$

$$\forall\, v \in H^3(I\!R^3) \cap \mathcal{E}, \|v\|_{H^3(I\!R^3)} \leq K \|L^{\frac{3}{2}} v\|_{L^2(I\!R^3)}.$$

Proof. From Proposition 3.1, there exists a constant K such that for $u \in H^2(I\!R)$, if $\displaystyle\int_{I\!R} u(x) \frac{1}{\mathrm{ch}\, x} dx = 0$, then

$$\|u\|^2_{L^2(I\!R)} + \|\partial_{xx} u\|^2_{L^2(I\!R)} \leq K \|L_1 u\|^2_{L^2(I\!R)}.$$

Now for $v \in \mathcal{E}$, we have for almost every $(y, z) \in I\!R^2$:

$$\int_{x \in I\!R} \left(|v(x, y, z)|^2 + |\partial_{xx} v(x, y, z)|^2 \right) dx \leq K \int_{I\!R} |L_1 v(x, y, z)|^2 dx.$$

So integrating for $(y, z) \in I\!R^2$ we obtain:

$$\|v\|^2_{L^2[I\!R^3)} + \|\partial_{xx} v\|^2_{L^2(I\!R^3)} \leq K \|L_1 v\|^2_{L^2(I\!R^3]}.$$

On the other hand,

$$\int_{I\!R^3} |Lv|^2 = \int_{I\!R^3} |L_1 v|^2 + \int_{I\!R^3} |\Delta_Y v|^2 - 2 \int_{I\!R^3} L_1 v \Delta_Y v,$$

where $\Delta_Y = \partial_{yy} + \partial_{zz}$. The last term is positive:

$$-2 \int_{I\!R^3} L_1 v \Delta_Y v = -2 \int_{I\!R^3} l^* \circ lv \cdot \Delta_Y v = 2 \int_{I\!R^3} |\nabla lv|^2,$$

by integrations by parts. So

$$\int_{I\!R^3} |Lv|^2 \geq \int_{I\!R^3} |L_1 v|^2 + \int_{I\!R^3} |\Delta_Y v|^2,$$

that is

$$\|v\|^2_{L^2(I\!R^3)} + \|\Delta v\|^2_{L^2(I\!R^3)} \leq K \|Lv\|^2_{L^2(I\!R^3)}.$$

The H^3 estimate can be proved with the same kind of arguments using Remark 3.1.

4 New coordinates

In the one dimensional case, i.e. for solutions depending only on the x-variable, we can construct a one parameter family of static solutions to the Landau-Lifschitz equation (1.5) using translational invariance. Indeed, for $s \in I\!R$, $x \mapsto M_0(x-s)$ satisfies (1.5). On the mobile frame, we consider the one parameter family $(R(s))_{s\in I\!R}$ of static solutions to (2.6) obtained from $M_0(x-s)$:

$$R(s)(x) = \begin{pmatrix} M_0(x-s)\cdot M_1(x) \\ M_0(x-s)\cdot M_2 \end{pmatrix} = \begin{pmatrix} \rho(s)(x) \\ 0 \end{pmatrix},$$

where $\rho(s)(x) = \dfrac{\operatorname{th} x}{\operatorname{ch}(x-s)} - \dfrac{\operatorname{th}(x-s)}{\operatorname{ch} x}$.

Following Kapitula [18], for r in a neighbourhood of 0, it would be desirable to use the coordinate system given by (σ, φ, W) with perturbations of zero being given by:

$$r(t,x,y,z) = R(\sigma(t,y,z))(x) + \begin{pmatrix} 0 \\ 1 \\ \hline \operatorname{ch} x \end{pmatrix} \varphi(t,y,z) + W(t,x,y,z), \tag{4.8}$$

where both coordinates of W take their values in $\mathcal{E}$. We prove that this system of coordinates is relevant in Proposition 4.1. To start with let us precise the notations.

We denote by Σ the following space

$$\Sigma = H^2(I\!R^2) \times H^2(I\!R^2) \times \mathcal{E} \times \mathcal{E}. \tag{4.9}$$

We endow Σ with the norm:

$$\|(\sigma,\varphi,W)\|_{\mathcal{H}^2} = \|\sigma\|_{H^2(I\!R^2)} + \|\varphi\|_{H^2(I\!R^2)} + \|LW_1\|_{L^2(I\!R^3)} + \|LW_2\|_{L^2(I\!R^3)}. \tag{4.10}$$

From Proposition 3.2, we have the following equivalence of norms on Σ:

$$\|(\sigma,\varphi,W)\|_{\mathcal{H}^2} \sim \|\sigma\|_{H^2(I\!R^2)} + \|\varphi\|_{H^2(I\!R^2)} + \|W_1\|_{H^2(I\!R^3)} + \|W_2\|_{H^2(I\!R^3)}.$$

In the same way, on $\Sigma \cap H^3$, we define

$$\|(\sigma,\varphi,W)\|_{\mathcal{H}^3} = \|\sigma\|_{H^3(I\!R^2)} + \|\varphi\|_{H^3(I\!R^2)} + \|L^{\frac{3}{2}}W_1\|_{L^2(I\!R^3)} + \|L^{\frac{3}{2}}W_2\|_{L^2(I\!R^3)}, \tag{4.11}$$

and this norm is equivalent to the H^3 norm on $\Sigma \cap H^3$:

$$\|(\sigma,\varphi,W)\|_{\mathcal{H}^3} \sim \|\sigma\|_{H^3(I\!R^2)} + \|\varphi\|_{H^3(I\!R^2)} + \|W_1\|_{H^3(I\!R^3)} + \|W_2\|_{H^3(I\!R^3)}.$$

Proposition 4.1. *There exists $\delta_0 > 0$, such that if $r \in H^2(I\!R^3; I\!R^2)$ satisfies $\|r\|_{H^2(I\!R^3)} \leq \delta_0$, there exists $(\sigma,\varphi,W) \in \Sigma$ such that*

$$r(x,y,z) = R(\sigma(y,z))(x) + \begin{pmatrix} 0 \\ 1 \\ \hline ch\, x \end{pmatrix} \varphi(y,z) + W(x,y,z).$$

In addition, there exists K such that for $r \in H^2(I\!R^3; I\!R^2)$ in a neighbourhood of zero,

$$\frac{1}{K}\|(\sigma,\varphi,W)\|_{\mathcal{H}^2} \leq \|r\|_{H^2(I\!R^3)} \leq K\|(\sigma,\varphi,W)\|_{\mathcal{H}^2}, \tag{4.12}$$

and for $r \in H^3(I\!R^3; I\!R^2)$ in a neighbourhood of zero,

$$\frac{1}{K}\|(\sigma,\varphi,W)\|_{\mathcal{H}^3} \leq \|r\|_{H^3(I\!R^3)} \leq K\|(\sigma,\varphi,W)\|_{\mathcal{H}^3}. \tag{4.13}$$

Proof. Let us introduce l^1 and l^2 defined for $r=(r_1,r_2)\in H^2(I\!\!R^3;I\!\!R^2)$ by:

$$l^1(r)(y,z)=\frac{1}{2}\int_{x\in I\!\!R} r_1(x,y,z)\frac{1}{\operatorname{ch} x}dx,\quad l^2(r)(y,z)=\frac{1}{2}\int_{x\in I\!\!R} r_2(x,y,z)\frac{1}{\operatorname{ch} x}dx.$$

The operators l^1 and l^2 are continuous linear mappings from $H^2(I\!\!R^3;I\!\!R^2)$ (resp. $H^3(I\!\!R^3;I\!\!R^2)$) into $H^2(I\!\!R^2)$ (resp. $H^3(I\!\!R^2)$).
Also we remark that $\mathcal{E}^2=\left\{W\in H^2(I\!\!R^3;I\!\!R^2), l^1(W)=l^2(W)=0\right\}$.

For a fixed r in a neigbourhood of 0, (σ,φ,W) can be found in the following manner:

- applying l^2 on (4.8) we obtain:
$$l^2(r)(y,z)=\varphi(y,z),$$
- applying l^1 on (4.8) yields:
$$l^1(r)=\frac{1}{2}\int_{x\in I\!\!R}\rho(\sigma(y,z))(x)\frac{1}{\operatorname{ch} x}dx.$$
Let us consider $\psi\in\mathcal{C}^\infty(I\!\!R;I\!\!R)$ given by
$$\psi(s)=\frac{1}{2}\int_{x\in I\!\!R}\rho(s)(x)\frac{1}{\operatorname{ch} x}dx.$$
Since $\psi(0)=0$ and $\psi'(0)=1$, there exists $\delta_0>0$ such that ψ is a $\mathcal{C}^\infty$-diffeomorphism from $]-\delta_0,\delta_0[$ to a neighbourhood of zero. We obtained
$$l^1(r)(y,z)=\psi(\sigma(y,z)),$$
so σ is given by
$$\sigma(y,z)=\psi^{-1}(l^1(r)(y,z)).$$
- By subtraction, we set
$$W(x,y,z)=r(x,y,z)-R(\sigma(y,z))(x)-\begin{pmatrix}0\\ \frac{1}{\operatorname{ch} x}\end{pmatrix}\varphi(y,z),$$
and by construction $l^1(W)=l^2(W)=0$, that is $W\in\mathcal{E}^2$.

Concerning (4.12), with straighforward estimates, using that $\rho(0)(x)=1$ and $\partial_s\rho(0)(x)=\frac{1}{\operatorname{ch} x}$ we obtain for example that for $\sigma\in H^2(I\!\!R^3)$ sufficiently small

$$\|(x,y,z)\mapsto R(\sigma(y,z))(x)\|_{H^2(I\!\!R^3)}\le K\|\sigma\|_{H^2(I\!\!R^2)},$$

so

$$\|r\|_{H^2(I\!\!R^3)}\le K\left(\|\sigma\|_{H^2(I\!\!R^2)}+\|\varphi\|_{H^2(I\!\!R^2)}+\|W\|_{H^2(I\!\!R^3)}\right)\le K\|(\sigma,\varphi,W)\|_{\mathcal{H}^2}.$$

By the continuity of the linear operators l^1 and l^2 for the H^2 norm, since ψ^{-1} is smooth in a neighbourhood of 0 and satisfies $\psi^{-1}(s)=s+\mathcal{O}(s^2)$, we obtain that

$$\|\sigma\|_{H^2(I\!\!R^2)}+\|\varphi\|_{H^2(I\!\!R^2)}\le K\|r\|_{H^2(I\!\!R^3)},$$

and by difference we obtain the claimed estimate on W. We prove (4.13) in the same way. This concludes the proof of Proposition 4.1.

Therefore in a neighbourhood of zero, we describe r in the coordinates (σ,φ,W) given by (4.8). Let us rewrite (2.6) in these coordinates. We assume that δ_0 is small enough to ensure that $\|r\|_{L^\infty}<1$, so that (2.6) makes sense.

We first remark that in the one dimensional case, for a fixed s, the map $x \mapsto R(s)(x)$ is a static solution to (2.6). So denoting by Λ_1 the reduced operator:

$$\Lambda_1 w = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} L_1 w_1 \\ L_1 w_2 + w_2 \end{pmatrix},$$

we have

$$\Lambda_1 R(\sigma) + A(R(\sigma))\partial_{xx} R(\sigma) + B(R(\sigma))(\partial_x R(\sigma), \partial_x R(\sigma)) + C(R(\sigma))(\partial_x R(\sigma)) + D(R(\sigma)) = 0. \quad (4.14)$$

Furthermore,

$$\partial_t (R(\sigma(t,y,z))(x) = \partial_s R(\sigma(t,y,z))\partial_t \sigma(t,y,z),$$

and

$$\Delta(R(\sigma(t,y,z))(x)) = \partial_{xx} R(\sigma(t,y,z)) + \partial_s R(\sigma(t,y,z))(\Delta_Y \sigma) + \partial_{ss} R(\sigma(t,y,z))|\nabla_Y \sigma|^2,$$

with $\Delta_Y := \partial_{yy} + \partial_{zz}$ and $|\nabla_Y \sigma|^2 := |\partial_y \sigma|^2 + |\partial_z \sigma|^2$. So, we have:

$$\Lambda R(\sigma) = \Lambda_1 R(\sigma) + \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} (-\partial_s R(\sigma)\Delta_Y \sigma - \partial_{ss} R(\sigma)|\nabla_Y \sigma|^2).$$

Plugging (4.8) in (2.6) and using (4.14) yield:

$$\begin{aligned} \partial_s R(\sigma)\partial_t \sigma + \begin{pmatrix} 0 \\ \dfrac{1}{\operatorname{ch} x} \end{pmatrix} \partial_t \varphi + \partial_t W = & \left(\partial_s \rho(\sigma)\Delta_Y \sigma - \partial_{ss}\rho(\sigma)|\nabla_Y \sigma|^2\right) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ & + \frac{1}{\operatorname{ch} x}(-\Delta_Y \varphi + \varphi) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \Lambda W + G. \end{aligned} \quad (4.15)$$

The nonlinear term G is defined by

$$G = G_1 + G_2 + \ldots + G_5, \quad (4.16)$$

where

- $G_1 = A(R(\sigma))\Delta_Y R(\sigma) + \tilde{A}(R(\sigma), w)(w)(\Delta r) + A(r)\Delta w,$
- $G_2 = 2B(R(\sigma))(\partial_x R(\sigma), \partial_x w) + B(R(\sigma))(\partial_x w, \partial_x w) + \tilde{B}(R(\sigma), w)(w))(\partial_x r, \partial_x r),$
- $G_3 = \displaystyle\sum_{i=2}^{3} B(r)(\partial_i r, \partial_i r),$
- $G_4 = C(x, R(\sigma))(\partial_x w) + \tilde{C}(x, R(\sigma), w)(w)(\partial_x r),$
- $G_5 = \tilde{D}(x, R(\sigma), w)(w),$

with the following notations:

- $w = \varphi(x) \begin{pmatrix} 0 \\ \dfrac{1}{\operatorname{ch} x} \end{pmatrix} + W$ and $r = R(\sigma) + w,$
- $\tilde{A} \in \mathcal{C}^\infty(B(0,1/2) \times B(0,1/2); \mathcal{L}(I\!R^2; \mathcal{M}_2(I\!R)))$:
$$\tilde{A}(u,v) = \int_0^1 A'(u + sv) ds,$$
- $\tilde{B} \in \mathcal{C}^\infty(B(0,1/2) \times B(0,1/2); \mathcal{L}(I\!R^2; \mathcal{L}_2(I\!R^2; I\!R^2)))$:
$$\tilde{B}(u,v) = \int_0^1 B'(u + sv) ds,$$

- $\tilde{C} \in \mathcal{C}^\infty(B(0,1/2)\times B(0,1/2);\mathcal{L}(I\!R^2;\mathcal{M}_2(I\!R)))$:

$$\tilde{C}(x,u,v) = \int_0^1 \partial_r C(x,u+sv)ds,$$

- $\tilde{D} \in \mathcal{C}^\infty(B(0,1/2)\times B(0,1/2);\mathcal{L}(I\!R^2;I\!R^2))$:

$$\tilde{D}(x,u,v) = \int_0^1 \partial_\xi D(x,u+sv)ds$$

(the tilda terms come from the fundamental theorem of the analysis applied between $R(\sigma)$ and $R(\sigma)+w$).

In order to separate the unknowns, we will use the projectors l^1 and l^2.

We multiply (4.15) by $\begin{pmatrix} \frac{1}{2\mathrm{ch}\,x} \\ 0 \end{pmatrix}$ and we integrate in the x variable. We obtain:

$$\tilde{g}(\sigma)\partial_t\sigma = \tilde{g}(\sigma)\Delta_Y\sigma + \Delta_Y\varphi - \varphi + \tilde{K}(\sigma)|\nabla_Y\sigma|^2 + l^1(G),$$

where

$$\tilde{g}(s) = \frac{1}{2}\int_{I\!R}\partial_s\rho(s)(x)\frac{1}{\mathrm{ch}\,x}dx = \frac{1}{2}\int_{I\!R}\left[\frac{\mathrm{sh}\,(x-s)\mathrm{th}\,x}{\mathrm{ch}^2(x-s)} + \frac{1}{\mathrm{ch}^2(x-s)\mathrm{ch}\,x}\right]\frac{1}{\mathrm{ch}\,x}dx,$$

and

$$\begin{aligned}\tilde{K}(s) = &\ \frac{1}{2}\int_{I\!R}\partial_{ss}\rho(s)(x)\frac{1}{\mathrm{ch}\,x}dx \\ = &\ \int_{I\!R}\left[-\frac{\mathrm{th}\,x}{\mathrm{ch}\,(x-s)} + 2\frac{\mathrm{sh}^2(x-s)\mathrm{th}\,x}{\mathrm{ch}^3(x-s)} + 2\frac{\mathrm{sh}\,(x-s)}{\mathrm{ch}^3(x-s)\mathrm{ch}\,x}\right]\frac{1}{\mathrm{ch}\,x}dx.\end{aligned}$$

We remark that $\tilde{g}$ and $\tilde{K}$ are in $\mathcal{C}^\infty(I\!R;I\!R)$ and that $\tilde{g}(0)=1$ and $\tilde{K}(0)=0$.
Then we write $\frac{1}{\tilde{g}(s)} = 1+\gamma(s)$ where $\gamma(s) = \mathcal{O}(|s|)$ in a neighbourhood of zero. So we obtain that

$$\partial_t\sigma = \Delta_Y\sigma + \Delta_Y\varphi - \varphi + T_1(\sigma,\varphi,W), \tag{4.17}$$

with

$$T_1(\sigma,\varphi,W) = \gamma(\sigma)(\Delta_Y\varphi-\varphi) + \frac{\tilde{K}(\sigma)}{\tilde{g}(\sigma)}|\nabla_Y\sigma|^2 + \frac{1}{\tilde{g}(\sigma)}l^1(G). \tag{4.18}$$

Now we multiply (4.15) by $\begin{pmatrix} 0 \\ \frac{1}{2\mathrm{ch}\,x} \end{pmatrix}$ and we integrate in the x variable. We get:

$$\partial_t\varphi = -\Delta\sigma + \Delta\varphi - \varphi + T_2(\sigma,\varphi,W), \tag{4.19}$$

where

$$T_2(\sigma,\varphi,W) = (1-\tilde{g}(\sigma))\Delta_Y\sigma + \tilde{K}(\sigma)|\nabla_Y\sigma|^2 + l^2(G). \tag{4.20}$$

Multiplying (4.17) by $\partial_s R(\sigma)$, (4.19) by $\begin{pmatrix} 0 \\ \frac{1}{\mathrm{ch}\,x} \end{pmatrix}$ and subtracting from (4.15) yield:

$$\partial_t W = \Lambda W + T_3(x,\sigma,\varphi,W). \tag{4.21}$$

The nonlinear term T_3 reads

$$T_3(x,\sigma,\varphi,W) = G + \left(\begin{array}{c} -|\nabla_Y \sigma|^2 \partial_{ss}\rho(\sigma) + (\Delta_Y \varphi - \varphi)\left(\dfrac{1}{\operatorname{ch} x} - \partial_s \rho(\sigma)\right) - \rho(\sigma) T_1(\sigma,\varphi,W) \\ |\nabla_Y \sigma|^2 \partial_{ss}\rho(\sigma) + \Delta_Y \sigma \left(\dfrac{1}{\operatorname{ch} x} - \partial_s \rho(\sigma)\right) - \dfrac{1}{\operatorname{ch} x} T_2(\sigma\varphi, W) \end{array} \right). \tag{4.22}$$

We have proved the following proposition:

Proposition 4.2. *Let $r \in \mathcal{C}^1(0,T;H^2(I\!R^3;I\!R^2))$ such that for all $t \geq 0$, $\|r(t,.)\|_{H^2(I\!R^3)} \leq \delta_0$. Let $(\sigma,\varphi,W) \in \mathcal{C}^1(0,T;\Sigma)$ given by proposition (4.1). Then r satisfies (2.6) if and only if (σ,φ,W) satisfies the system (4.17)-(4.19)-(4.21), and 0 is stable for (2.6) if and only if $(0,0,0)$ is stable for (4.17)-(4.19)-(4.21).*

Remark 4.1. *The key point of this step is that with l^1 and l^2, we can separate the variables σ, φ and W in order to obtain the system (4.17)-(4.19)-(4.21) in which the linear parts are almost independent. When we deal with the complete model for the demagnetizing field or with the travelling waves solutions when a magnetic field is applied, this splitting is not possible and we are unable to perform successful variational estimates.*

5 Variational Estimates

We recall that we deal with the following system:

$$\partial_t \sigma = \Delta_Y \sigma + \Delta_Y \varphi - \varphi + T_1(\sigma,\varphi,W), \tag{5.23}$$

$$\partial_t \varphi = -\Delta\sigma + \Delta\varphi - \varphi + T_2(\sigma,\varphi,W), \tag{5.24}$$

$$\partial_t W = \left(\begin{array}{c} -LW_1 - (L+1)W_2 \\ LW_1 - (L+1)W_2 \end{array} \right) + T_3(x,\sigma,\varphi,W). \tag{5.25}$$

The unknown (σ,φ,W_1,W_2) takes its values in Σ defined in (4.9). The nonlinear terms T_1, T_2 and T_3 are defined in (4.18), (4.20) and (4.22) respectively.

Our stability result is similar to a global existence with small data theorem. By variational estimates we will prove that if the initial data are small then the solution of (5.23)-(5.24)-(5.25) remains small. When we multiply the equations by the unknowns or their space derivatives, the linear part yields good sign absorbing terms. In order to be able to absorb the nonlinear terms, we have to control them by the absorbing terms. We claim the following proposition:

Proposition 5.1. *There exists K such that for all $(\sigma,\varphi,W) \in \Sigma$, if $\|(\sigma,\varphi,W)\|_{\mathcal{H}^2} \leq \gamma_1$, then*

$$\begin{aligned} &\|T_1\|_{H^1(I\!R^2)} + \|T_2\|_{H^1(I\!R^2)} + \|T_3\|_{H^1(I\!R^3)} \\ &\qquad \leq K\|(\sigma,\varphi,W)\|_{\mathcal{H}^2} \left(\|\Delta_Y \sigma\|_{H^1(I\!R^2)} + \|\varphi\|_{H^3(I\!R^2)} + \|W\|_{H^3(I\!R^3)} \right). \end{aligned} \tag{5.26}$$

In addition, we can split $T_1 - T_2$ on the form : $T_1 - T_2 = \tilde{T}_a + \tilde{T}_b$, where $\tilde{T}_a$ and $\tilde{T}_b$ satisfy the following estimates: there exists K such that for all $(\sigma,\varphi,W) \in \Sigma$, if $\|(\sigma,\varphi,W)\|_{\mathcal{H}^2} \leq \gamma_1$, then

$$\left\{ \begin{array}{l} \|\tilde{T}_a\|_{L^1(I\!R^2)} \leq K \left(\|\nabla_Y \sigma\|_{H^2(I\!R^2)} + \|\varphi\|_{H^3(I\!R^2)} + \|W\|_{H^3(I\!R^3)} \right)^2, \\ \\ \|\tilde{T}_b\|_{L^{\frac{4}{3}}(I\!R^2)} \leq K \left(\|\nabla_Y \sigma\|_{H^2(I\!R^2)} + \|\varphi\|_{H^3(I\!R^2)} + \|W\|_{H^3(I\!R^3)} \right) \|\sigma\|_{L^4(I\!R^2)}. \end{array} \right. \tag{5.27}$$

For the convenience of the reader we postpone the proof of this proposition in the last section.

Before starting the variational estimates, we establish a Sobolev type inequality in 2d:

Lemma 5.1. *There exists a constant K such that for all $u \in H^2(I\!R^2)$,*

$$\|u\|_{L^4(I\!R^2)} \leq K\|u\|^{\frac{1}{2}}_{L^2(I\!R^2)}\|\nabla_Y u\|^{\frac{1}{2}}_{L^2(I\!R^2)}.$$

Proof: in the 2-dimensional case, from Sobolev imbeddings, $W^{1,1}(I\!R^2) \hookrightarrow L^2(I\!R^2)$ and there exists K such that

$$\|v\|_{L^2(I\!R^2)} \leq K\|\nabla_Y v\|_{L^1(I\!R^2)}.$$

We apply the previous inequality to u^2 to conclude the proof of Lemma 5.1.

5.1 H^1 and H^2 estimates

Taking the inner product of (5.23) with $-\Delta_Y\sigma$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla_Y\sigma\|^2_{L^2(I\!R^2)}\right) + \|\Delta_Y\sigma\|^2_{L^2(I\!R^2)} = -\int_{I\!R^2}(\Delta_Y\varphi - \varphi)\Delta_Y\sigma - \int_{I\!R^2} T_1(\sigma,\varphi,W)\Delta_Y\sigma.$$

Taking the inner product of (5.24) with $-\Delta_Y\varphi + \varphi$ we get:

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla_Y\varphi\|^2_{L^2(I\!R^2)} + \|\varphi\|^2_{L^2(I\!R^2)}\right) + \|\Delta_Y\varphi - \varphi\|^2_{L^2(I\!R^2)} = \int_{I\!R^2}(\Delta_Y\varphi - \varphi)\Delta_Y\sigma$$
$$- \int_{I\!R^2} T_2(\sigma,\varphi,W)(\Delta_Y\varphi - \varphi).$$

Adding the previous equations, we obtain:

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla_Y\sigma\|^2_{L^2(I\!R^2)} + \|\varphi\|^2_{L^2(I\!R^2)} + \|\nabla_Y\varphi\|^2_{L^2(I\!R^2)}\right) + \Big[\|\Delta_Y\sigma\|^2_{L^2(I\!R^2)} + \|\varphi\|^2_{L^2(I\!R^2)} + 2\|\nabla_Y\varphi\|^2_{L^2(I\!R^2)}$$
$$+\|\Delta_Y\varphi\|^2_{L^2(I\!R^2)}\Big] = -\int_{I\!R^2} T_1(\sigma,\varphi,W)\Delta_Y\sigma - \int_{I\!R^2} T_2(\sigma,\varphi,W)(\Delta_Y\varphi - \varphi). \tag{5.28}$$

Taking the inner product of (5.23) with $\Delta_Y^2\sigma$ and the product of (5.24) with $\Delta_Y(\Delta_Y\varphi - \varphi)$ yield:

$$\frac{1}{2}\frac{d}{dt}\left(\|\Delta_Y\sigma\|^2_{L^2(I\!R^2)} + \|\nabla_Y\varphi\|^2_{L^2(I\!R^2)} + \|\Delta_Y\varphi\|^2_{L^2(I\!R^2)}\right) + \Big[\|\nabla_Y\Delta_Y\sigma\|^2_{L^2(I\!R^2)} + \|\nabla_Y\varphi\|^2_{L^2(I\!R^2)}$$
$$+2\|\Delta_Y\varphi\|^2_{L^2(I\!R^2)} + \|\nabla_Y\Delta_Y\varphi\|^2_{L^2(I\!R^2)}\Big] = -\int_{I\!R^2}\nabla_Y(T_1(\sigma,\varphi,W))\cdot\nabla_Y\Delta_Y\sigma$$
$$-\int_{I\!R^2}\nabla_Y(T_2(\sigma,\varphi,W))\cdot\nabla_Y(\Delta_Y\varphi - \varphi). \tag{5.29}$$

Estimates 5.26 in Proposition 5.1 together with (5.28) and (5.29) yield that while $\|(\sigma,\varphi,W)\|_{\mathcal{H}^2} \leq \gamma_1$, then

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla_Y\sigma\|^2_{L^2(I\!R^2)} + \|\Delta_Y\sigma\|^2_{L^2(I\!R^2)} + \|\varphi\|^2_{L^2(I\!R^2)} + 2\|\nabla_Y\varphi\|^2_{L^2(I\!R^2)} + \|\Delta_Y\varphi\|^2_{L^2(I\!R^2)}\right)$$
$$+\Big[\|\Delta_Y\sigma\|^2_{L^2(I\!R^2)} + \|\nabla_Y\Delta_Y\sigma\|^2_{L^2(I\!R^2)} + \|\varphi\|^2_{L^2(I\!R^2)} + 3\|\nabla_Y\varphi\|^2_{L^2(I\!R^2)} + 3\|\Delta_Y\varphi\|^2_{L^2(I\!R^2)} \tag{5.30}$$
$$+\|\nabla_Y\Delta_Y\varphi\|^2_{L^2(I\!R^2)}\Big] \leq K\|(\sigma,\varphi,W)\|_{\mathcal{H}^2}\left(\|\Delta_Y\varphi\|^2_{H^1(I\!R^2)} + \|\varphi\|^2_{H^3(I\!R^2)} + \|W\|^2_{H^3(I\!R^3)}\right).$$

Taking the inner product of (5.25) with $\begin{pmatrix} L^2 W_1 \\ L(L+1)W_2 \end{pmatrix}$ yields:

$$\frac{1}{2}\frac{d}{dt}\left(\|LW_1\|^2_{L^2(I\!R^3)} + \|(L+Id)W_2\|^2_{L^2(I\!R^3)}\right) + \|L^{\frac{3}{2}}W_1\|^2_{L^2(I\!R^3)} + \|L^{\frac{1}{2}}(L+Id)W_2\|^2_{L^2(I\!R^3)}$$

$$\le \|L^{\frac{1}{2}}T_3\|_{L^2(I\!R^3)}\left(\|L^{\frac{3}{2}}W_1\|_{L^2(I\!R^3)} + \|L^{\frac{1}{2}}(L+Id)W_2\|_{L^2(I\!R^3)}\right) \tag{5.31}$$

$$\le K\|(\sigma,\varphi,W)\|_{\mathcal{H}^2}\left(\|\Delta_Y\varphi\|^2_{H^1(I\!R^2)} + \|\varphi\|^2_{H^3(I\!R^2)} + \|W\|^2_{H^3(I\!R^3)}\right).$$

while $\|(\sigma,\varphi,W)\|_{\mathcal{H}^2} \le \gamma_1$ (by Proposition 5.1).

5.2 L^2-estimates

Subtracting (5.23) to (5.24) yields

$$\partial_t(\sigma-\varphi) = 2\Delta_Y\sigma + T_1(\sigma,\varphi,W) - T_2(\sigma,\varphi,W).$$

Multiplying by $\sigma-\varphi$, we obtain:

$$\frac{1}{2}\frac{d}{dt}\|\sigma-\varphi\|^2_{L^2(I\!R^2)} + 2\|\nabla_Y\sigma\|^2_{L^2(I\!R^2)} = 2\int_{I\!R^2}\nabla_Y\sigma\nabla_Y\varphi + \int_{I\!R^2}(T_1-T_2)\sigma - \int_{I\!R^2}(T_1-T_2)\varphi.$$

By Young inequality and with the splitting of T_1-T_2 (see Proposition 5.1), we have

$$\frac{1}{2}\frac{d}{dt}\left(\|\sigma-\varphi\|^2_{L^2(I\!R^2)}\right) + 2\|\nabla_Y\sigma\|^2_{L^2(I\!R^2)} \le \|\nabla_Y\sigma\|^2_{L^2(I\!R^2)} + \|\nabla_Y\varphi\|^2_{L^2(I\!R^2)} + \|\tilde{T}_a\|_{L^1(I\!R^2)}\|\sigma\|_{L^\infty(I\!R^2)}$$

$$+\|\tilde{T}_b\|_{L^{\frac{4}{3}}(I\!R^2)}\|\sigma\|_{L^4(I\!R^2)} + (\|T_1\|_{L^2(I\!R^2)} + \|T_2\|_{L^2(I\!R^2)})\|\varphi\|_{L^2(I\!R^2)}.$$

So, applying Estimate (5.27) (see Proposition 5.1), while $\|(\sigma,\varphi,W)\|_{\mathcal{H}^2} \le \gamma_1$, we get

$$\frac{1}{2}\frac{d}{dt}\left(\|\sigma-\varphi\|^2_{L^2(I\!R^2)}\right) + \|\nabla_Y\sigma\|^2_{L^2(I\!R^2)} \le \|\nabla_Y\varphi\|^2_{L^2(I\!R^2)}$$

$$+K\|\sigma\|_{L^\infty(I\!R^2)}\left[\|\nabla_Y\sigma\|_{H^2(I\!R^2)} + \|\varphi\|_{H^3(I\!R^2)} + \|W\|_{H^3(I\!R^3)}\right]^2$$

$$+K\left[\|\nabla_Y\sigma\|_{H^2(I\!R^2)} + \|\varphi\|_{H^3(I\!R^2)} + \|W\|_{H^3(I\!R^3)}\right]\|\sigma\|^2_{L^4(I\!R^2)}$$

$$+K\|(\sigma,\varphi,W)\|_{\mathcal{H}^2}\left[\|\Delta_Y\sigma\|_{L^2(I\!R^2)} + \|\varphi\|_{H^2(I\!R^2)} + \|W\|_{H^2(I\!R^3)}\right]\|\varphi\|_{L^2(I\!R^2)}.$$

By Lemma 5.1,

$$\|\sigma\|^2_{L^4(I\!R^2)} \le K\|\sigma\|_{L^2(I\!R^2)}\|\nabla_Y\sigma\|_{L^2(I\!R^2)}.$$

So, we obtain that while $\|(\sigma,\varphi,W)\|_{\mathcal{H}^2} \le \gamma_1$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|\sigma-\varphi\|^2_{L^2(I\!R^2)}\right) + \|\nabla_Y\sigma\|^2_{L^2(I\!R^2)} &\le \|\nabla_Y\varphi\|^2_{L^2(I\!R^2)} \\ &+K\|(\sigma,\varphi,W)\|_{\mathcal{H}^2)}\left[\|\nabla_Y\sigma\|^2_{L^2(I\!R^2)} + \|\varphi\|^2_{H^3(I\!R^2)} + \|W\|^2_{H^3(I\!R^3)}\right].\end{aligned} \tag{5.32}$$

5.3 End of the proof

We define $\mathcal{N}$ and $\mathcal{D}$ by

$$\mathcal{N}(t) = \Big(\|\sigma-\varphi\|^2_{L^2(I\!R^2)} + \|\nabla_Y\sigma\|^2_{L^2(I\!R^2)} + \|\Delta_Y\sigma\|^2_{L^2(I\!R^2)} + \|\varphi\|^2_{L^2(I\!R^2)} + 2\|\nabla_Y\varphi\|^2_{L^2(I\!R^2)}$$

$$+\|\Delta_Y\varphi\|^2_{L^2(I\!R^2)} + \|LW_1\|^2_{L^2(I\!R^3)} + \|(L+Id)W_2\|^2_{L^2(I\!R^3)}\Big)(t),$$

and

$$\mathcal{D}(t) = \Big[\|\nabla_Y \sigma\|^2_{L^2(I\!R^2)} + \|\Delta_Y \sigma\|^2_{L^2(I\!R^2)} + \|\nabla_Y \Delta_Y \sigma\|^2_{L^2(I\!R^2)} + \|\varphi\|^2_{L^2(I\!R^2)} + 2\|\nabla_Y \varphi\|^2_{L^2(I\!R^2)}$$

$$+3\|\Delta_Y \varphi\|^2_{L^2(I\!R^2)} + \|\nabla_Y \Delta_Y \varphi\|^2_{L^2(I\!R^2)} + \|L^{\frac{3}{2}} W_1\|^2_{L^2(I\!R^3)} + \|L^{\frac{1}{2}}(L+Id)W_2\|^2_{L^2(I\!R^3)} \Big](t).$$

Adding up (5.30), (5.31) and (5.32), we obtain that

$$\frac{1}{2}\frac{d\mathcal{N}}{dt} + \mathcal{D}(t) \leq K\|(\sigma,\varphi,W)\|_{\mathcal{H}^2} \left[\|\nabla_Y \sigma\|^2_{H^2(I\!R^2)} + \|\varphi\|^2_{H^3(I\!R^2)} + \|W\|^2_{H^3(I\!R^3)} \right]$$

(the term $\|\nabla_Y \varphi\|^2_{L^2(I\!R^2)}$ in the right hand side of (5.32) vanishes with a part of the left hand side of (5.30)).

As remarked in Proposition 3.2, on $\mathcal{E}$, we have the equivalences of norms: $\|L^{\frac{3}{2}} W_1\|_{L^2(I\!R^3)} \sim \|W_1\|_{H^3(I\!R^3)}$ and $\|L^{\frac{1}{2}}(L+Id)W_2\|_{L^2(I\!R^3)} \sim \|W_2\|_{H^3(I\!R^3)}$. So there exists a constant C_1 such that

$$\mathcal{D} \geq C_1 \left[\|\nabla_Y \sigma\|^2_{H^2(I\!R^2)} + \|\varphi\|^2_{H^3(I\!R^2)} + \|W\|^2_{H^3(I\!R^3)} \right].$$

In addition, $\|\sigma\|_{L^2(I\!R^2)} \leq \|\sigma - \varphi\|_{L^2(I\!R^2)} + \|\varphi\|_{L^2(I\!R^2)}$, so again with Proposition 3.2, there exists C_2 such that

$$\frac{1}{C_2}\|(\sigma,\varphi,W)\|_{\mathcal{H}^2} \leq \mathcal{N}(t) \leq C_2\|(\sigma,\varphi,W)\|_{\mathcal{H}^2}.$$

Hence while $\|(\sigma,\varphi,W)\|_{\mathcal{H}^2} \leq \gamma_1$, we have

$$\frac{1}{2}\frac{d\mathcal{N}}{dt} + \left[\|\nabla_Y \sigma\|^2_{H^2(I\!R^2)} + \|\varphi\|^2_{H^3(I\!R^2)} + \|W\|^2_{H^3(I\!R^3)} \right] (C_1 - KC_2\mathcal{N}(t)) \leq 0. \tag{5.33}$$

Let us introduce $\eta_0 = \min\left\{ \dfrac{\gamma_1}{C_2}, \dfrac{C_1}{KC_2} \right\}$. If $\mathcal{N}(0) \leq \eta_0$, then with (5.33), $\mathcal{N}(t)$ remains smaller than $\dfrac{C_1}{KC_2}$, that is $\mathcal{N}(t)$ decreases and remains smaller than η_0, so that $\|(\sigma,\varphi,W)\|_{\mathcal{H}^2}$ remains smaller than γ_1. So we are always in the validity domain of our estimates.

Therefore we have proved the stability of $(0,0,0)$ for (4.17)-(4.19)-(4.21). This concludes the proof of Theorem 1.1 using Propositions 2.1 and 4.2.

6 Proof of Proposition 5.1

We recall that from Proposition 4.1, for $r \in H^2(I\!R^3)$ in a neighbourhood of 0, we can write

$$r(x,y,z) = R(\sigma(y,z))(x) + \varphi(y,z)\begin{pmatrix} 0 \\ \frac{1}{\operatorname{ch} x} \end{pmatrix} + W(x,y,z),$$

with $(\sigma,\varphi,W) \in \Sigma$, and there exists K independant of r such that for $k = 2$ or 3,

$$\frac{1}{K}\|(\sigma,\varphi,W)\|_{\mathcal{H}^k} \leq \|r\|_{H^k(I\!R^3)} \leq K\|(\sigma,\varphi,W)\|_{\mathcal{H}^k}$$

(see (4.9), (4.10) and (4.11) for the notations).

We introduce $\gamma_1 > 0$ such that if $\|(\sigma,\varphi,W)\|_{\mathcal{H}^2} \leq \gamma_1$, then $\|r\|_{L^\infty} \leq \delta_0$, so that we are in the framework of Proposition 4.2.

To start with, we recall Gagliardo-Nirenberg type inequalities.

Lemma 6.1. *There exists a constant K such that for all $u \in H^2(I\!R^2)$,*

$$\|\nabla_Y u\|^2_{L^{2p}(I\!R^2)} \leq K\|u\|_{L^\infty(I\!R^2)}\|\Delta_Y u\|_{L^p(I\!R^2)} \text{ for } p = 1,2,4.$$

Proof. For $i \in \{2,3\}$ and for $p = 1,2,4$, we have:

$$\begin{aligned}\int_{\mathbb{R}^2} (\partial_i u)^{2p} &= \int_{\mathbb{R}^2} \partial_i u (\partial_i u)^{2p-1} \\ &= -(2p-1) \int_{\mathbb{R}^2} u \partial_{ii} u (\partial_i u)^{2p-2} \\ &\leq K \|u\|_{L^\infty(\mathbb{R}^2)} \|\partial_{ii} u\|_{L^p(\mathbb{R}^2)} \|\partial_i u\|_{L^{2p}(\mathbb{R}^2)}^{2p-2},\end{aligned}$$

which concludes the proof of Lemma 6.1.

6.1 Proof of Estimate (5.26)

In the following proposition, we estimate the nonlinear term G defined in (4.16) (we recall that this term appears in (4.15)).

Proposition 6.1. *There exists K such that for all $(\sigma, \varphi, W) \in \Sigma$, if $\|(\sigma, \varphi, W)\|_{\mathcal{H}^2} \leq \gamma_1$, then*

$$\|G\|_{L^2(\mathbb{R}^3)} + \|\nabla G\|_{L^2(\mathbb{R}^3)} \leq K \|(\sigma, \varphi, W)\|_{\mathcal{H}^2} \left(\|\Delta_Y\|_{H^1(\mathbb{R}^2)} + \|\varphi\|_{H^3(\mathbb{R}^2)} + \|W\|_{H^3(\mathbb{R}^3)} \right).$$

First we establish preliminary estimates.

Lemma 6.2. *There exists K such that for all $(\sigma, \varphi, W) \in \Sigma$, if $\|(\sigma, \varphi, W)\|_{\mathcal{H}^2} \leq \gamma_1$, then*

$$\|R(\sigma)\|_{L^\infty(\mathbb{R}^3)} + \|\nabla R(\sigma)\|_{L^4(\mathbb{R}^3)} + \|\nabla \partial_x R(\sigma)\|_{L^4(\mathbb{R}^3)} \leq K \|(\sigma, \varphi, W)\|_{\mathcal{H}^2},$$

and

$$\begin{aligned}&\|\Delta_Y R(\sigma)\|_{L^2(\mathbb{R}^3)} + \|\Delta_Y R(\sigma)\|_{L^4(\mathbb{R}^3)} + \|\nabla \Delta_Y R(\sigma)\|_{L^2(\mathbb{R}^3)} \\ &\leq K \left(\|\Delta_Y \sigma\|_{H^1(\mathbb{R}^2)} + \|\varphi\|_{H^3(\mathbb{R}^2)} + \|W\|_{H^3(\mathbb{R}^3)} \right).\end{aligned}$$

Proof. We recall that there exists K such that for s in the neighbourhood of 0, we have

- $|R(s)(x)| + |\partial_x R(s)(x)| + |\partial_{xx} R(s)(x)| \leq K \dfrac{|s|}{\operatorname{ch} x}$,
- $|\partial_s R(s)(x)| + |\partial_x \partial_s R(s)| \leq \dfrac{K}{\operatorname{ch} x}$,
- $|\partial_{ss} R(s)(x)| + |\partial_x \partial_{ss} R(s)(x)| \leq \dfrac{K}{\operatorname{ch} x}$,
- $|\partial_{sss} R(s)(x)| \leq \dfrac{K}{\operatorname{ch} x}$.

On one hand, the first claimed estimate is a straightforward consequence of the previous remarks and the Sobolev embeddings of $H^2(\mathbb{R}^2)$ into $L^\infty(\mathbb{R}^2)$ and $W^{1,4}(\mathbb{R}^2)$.

On the other hand,

$$\Delta_Y (R(\sigma)) = \partial_s R(\sigma) \Delta_Y \sigma + \partial_{ss} R(\sigma) |\nabla_Y \sigma|^2,$$

so

$$|\Delta_Y (R(\sigma))| \leq K (|\Delta_Y \sigma| + |\nabla_Y \sigma|^2) \frac{1}{\operatorname{ch} x}.$$

With Lemma 1,

$$\|\Delta_Y R(\sigma)\|_{L^2(\mathbb{R}^3)} \leq K \|\Delta_Y \sigma\|_{L^2(\mathbb{R}^2)}.$$

In addition,

$$\begin{aligned}\|\Delta_Y R(\sigma)\|_{L^4(\mathbb{R}^3)} &\leq K \|\Delta_Y \sigma\|_{L^4(\mathbb{R}^2)} + K \|\nabla_Y \sigma\|_{L^8(\mathbb{R}^2)}^2 \\ &\leq K \|\Delta_Y \sigma\|_{L^4(\mathbb{R}^2)} \text{ by Lemma 6.1,} \\ &\leq K \|\nabla_Y \sigma\|_{H^1(\mathbb{R}^2)} \text{ by Sobolev embedding.}\end{aligned}$$

To conclude, we have

$$\partial_x \Delta_Y R(\sigma) = \partial_x \partial_s R(\sigma)\Delta_Y \sigma + \partial_x \partial_{ss} R(\sigma)|\nabla_Y \sigma|^2,$$

so the estimate on $\partial_x \Delta_Y R(\sigma)$ is straightforward.
Concerning the derivatives in y and z, we have

$$\begin{aligned}\nabla_Y \Delta_Y R(\sigma) = \quad & \partial_{ss} R(\sigma)(\nabla_Y \sigma)\Delta_Y \sigma + \partial_s R(\sigma)\nabla_Y \Delta_Y \sigma + \partial_{sss} R(\sigma)(\nabla_Y \sigma)|\nabla_Y \sigma|^2 \\ & + 2\partial_{ss} R(\sigma)\nabla_Y^2 \sigma \cdot \nabla_Y \sigma,\end{aligned}$$

so

$$\begin{aligned}\|\nabla_Y \Delta_Y R(\sigma)\|_{L^2(I\!R^3)} \leq \quad & K\|\nabla_Y \sigma\|_{L^4(I\!R^2)}\|\Delta_Y \sigma\|_{L^4(I\!R^2)} + K\|\nabla_Y \Delta_Y \sigma\|_{L^2(I\!R^2)} + K\|\nabla_Y \sigma\|^3_{L^6(I\!R^2)} \\ & + K\|\nabla_Y^2 \sigma\|_{L^4(I\!R^2)}\|\nabla_Y \sigma\|_{L^4(I\!R^2)} \\ \leq \quad & K\left(\|\nabla_Y^2 \sigma\|_{L^2(I\!R^2)} + \|\nabla_Y^3 \sigma\|_{L^2(I\!R^2)}\right) \\ \leq \quad & \|\nabla_Y \sigma\|_{H^1(I\!R^2)}.\end{aligned}$$

This concludes the proof of Lemma 6.2.

We recall that we denote by w the quantity

$$w(t,x,y,z) = \varphi(t,x,y,z)\begin{pmatrix} 0 \\ 1 \\ \dfrac{}{\operatorname{ch} x} \end{pmatrix} + W(t,x,y,z).$$

Lemma 6.3. *There exists a constant K such that*

$$\|w\|_{L^\infty(I\!R^3)} + \|w\|_{H^2(I\!R^3)} + \|\nabla w\|_{L^4(I\!R^3)} \leq K\|(\sigma,\varphi,W)\|_{\mathcal{H}^2},$$

and

$$\|w\|_{H^2(I\!R^3)} + \|\Delta w\|_{L^4(I\!R^3)} + \|\nabla \Delta w\|_{L^2(I\!R^3)} \leq K\left(\|\Delta_Y\|_{H^1(I\!R^2)} + \|\varphi\|_{H^3(I\!R^2)} + \|W\|_{H^3(I\!R^3)}\right).$$

Proof. This lemma is a direct consequence of the Sobolev inequalities.

Proof of Proposition 6.1. We estimate each term of G separately (see (4.16)).

- We recall that

  $G_1 = A(R(\sigma))\Delta_Y R(\sigma) + \tilde{A}(R(\sigma),w)(w)(\partial_{xx} R(\sigma)) + \tilde{A}(R(\sigma,w)(w)\Delta_Y R(\sigma) + A(R(\sigma)+w)\Delta w.$

  In addition from proposition 2.2, there exists K such that for $|\xi| \leq \dfrac{1}{2}$,

  $$|A(\xi)| \leq K|\xi|, \; |A'(\xi)| \leq K,$$
  $$\tilde{A}(u,v) \leq K(|u|+|v|) \text{ and } |\partial_u \tilde{A}(u,v)| + |\partial_v \tilde{A}(u,v)| \leq K.$$

  Therefore

  $$|G_1| \leq K|R(\sigma)||\Delta_Y R(\sigma)| + K|w||\partial_{xx} R(\sigma)| + K|w||\Delta_Y R(\sigma)| + (|R(\sigma)| + |w|)|\Delta w|,$$

  so that

  $$\begin{aligned}\|G_1\|_{L^2(I\!R^3)} \leq \quad & K\left(\|R(\sigma)\|_{L^\infty(I\!R^3)} + \|w\|_{L^\infty(I\!R^3)}\right)\left(\|\Delta_Y R(\sigma)\| + \|\Delta w\|_{L^2(I\!R^3)}\right) \\ & + K\|\partial_{xx} R(\sigma)\|_{L^\infty(I\!R^3)}\|w\|_{L^2(I\!R^3)} \\ \leq \quad & K\|(\sigma,\varphi,W)\|_{\mathcal{H}^2}\left(\|\Delta_Y\|_{H^1(I\!R^2)} + \|\varphi\|_{H^3(I\!R^2)} + \|W\|_{H^3(I\!R^3)}\right),\end{aligned}$$

from Lemma 6.2 and Lemma 6.3.

Concerning the gradient we have

$$\begin{aligned}|\nabla G_1| \leq \quad & K|\nabla R(\sigma)||\Delta_Y R(\sigma)| + K|R(\sigma)||\nabla\Delta_Y R(\sigma)| + K\left(|\nabla R(\sigma)| + |\nabla w|\right)|w||\partial_{xx}R(\sigma)| \\ & + K|\nabla w||\partial_{xx}R(\sigma)| + K\left(|\nabla R(\sigma)| + |\nabla w|\right)|w||\Delta_Y R(\sigma)| \\ & + K|\nabla w||\Delta_Y R(\sigma)| + \left(|\nabla R(\sigma)| + |\nabla w|\right)|\Delta w| + K\left(|R(\sigma)| + |w|\right)|\nabla\Delta w|.\end{aligned}$$

Thus

$$\begin{aligned}\|\nabla G_1\|_{L^2(\mathbb{R}^3)} \leq \quad & K\|\nabla R(\sigma)\|_{L^4(\mathbb{R}^3)}\|\Delta_Y R(\sigma)\|_{L^4(\mathbb{R}^3)} + K\|R(\sigma)\|_{L^\infty(\mathbb{R}^3)}\|\nabla\Delta_Y R(\sigma)\|_{L^2(\mathbb{R}^3)} \\ & + K\left(\|\nabla R(\sigma)\|_{L^4(\mathbb{R}^3)} + \|\nabla w\|_{L^4(\mathbb{R}^3)}\right)\|w\|_{L^4(\mathbb{R}^3)}\|\partial_{xx}R(\sigma)\|_{L^\infty(\mathbb{R}^3)} \\ & + K\|\nabla w\|_{L^2(\mathbb{R}^3)}\|\partial_{xx}R(\sigma)\|_{L^\infty(\mathbb{R}^3)} \\ & + K\left(\|\nabla R(\sigma)\|_{L^4(\mathbb{R}^3)} + \|\nabla w\|_{L^4(\mathbb{R}^3)}\right)\|w\|_{L^\infty(\mathbb{R}^3)}\|\Delta_Y R(\sigma)\|_{L^4(\mathbb{R}^3)} \\ & + K\|\nabla w\|_{L^4(\mathbb{R}^3)}\|\Delta_Y R(\sigma)\|_{L^4(\mathbb{R}^3)} \\ & + \left(\|\nabla R(\sigma)\|_{L^4(\mathbb{R}^3)} + \|\nabla w\|_{L^4(\mathbb{R}^3)}\right)\|\Delta w\|_{L^4(\mathbb{R}^3)} \\ & + K\left(\|R(\sigma)\|_{L^\infty(\mathbb{R}^3)} + \|w\|_{L^\infty(\mathbb{R}^3)}\right)\|\nabla\Delta w\|_{L^2(\mathbb{R}^3)} \\ \leq \quad & K\|(\sigma,\varphi,W)\|_{\mathcal{H}^2}\left(\|\Delta_Y\|_{H^1(\mathbb{R}^2)} + \|\varphi\|_{H^3(\mathbb{R}^2)} + \|W\|_{H^3(\mathbb{R}^3)}\right),\end{aligned}$$

using Lemmas 6.2 and 6.3.

- We have

$$\begin{aligned}G_2 = \quad & 2B(R(\sigma))(\partial_x R(\sigma), \partial_x w) + B(R(\sigma))(\partial_x w, \partial_x w) + \tilde{B}(R(\sigma, w)(w)(\partial_x R(\sigma), \partial_x R(\sigma)) \\ & + 2\tilde{B}(R(\sigma, w)(w)(\partial_x R(\sigma), \partial_x w) + \tilde{B}(R(\sigma, w)(w)(\partial_x w, \partial_x w).\end{aligned}$$

  Furthermore, we recall that from Proposition 2.2, there exists K such that for $|\xi| \leq \frac{1}{2}$ one has

$$|B(\xi)| \leq K|\xi|, \; |B'(\xi)| \leq K,$$

  and for $|u| \leq 1/2$ and $|v| \leq 1/2$,

$$|\tilde{B}(u,v)| + |\partial_u \tilde{B}(u,v)| + |\partial_v \tilde{B}(u,v)| \leq K.$$

  A straightforward calculation, Lemma 6.2 and Lemma 6.3 yield the expected estimates on G_2 and ∇G_2.

- The term G_3 is given by

$$G_3 = B(r)(\partial_s R(\sigma), \partial_s R(\sigma))|\nabla\sigma|^2 + 2\sum_{i=2}^{3} B(r)(\partial_s R(\sigma), \partial_i w)\partial_i\sigma + \sum_{i=2}^{3} B(r)(\partial_i w, \partial_i w).$$

  Using that $|B(\xi)| \leq K|\xi|$ and that $|B'(\xi)| \leq K$ for $\xi \in B(0,1/2)$, since, by Lemma 6.1, $\|\nabla_Y\sigma\|_{L^4(\mathbb{R}^2)} \leq K\|\sigma\|_{L^\infty(\mathbb{R}^2)}\|\Delta_Y\sigma\|_{L^2(\mathbb{R}^2)}$, we obtain the claimed estimate on G_3.

- To estimate G_4, we remark that

$$G_4 = C(x, R(\sigma))(\partial_x w) + \tilde{C}(x, R(\sigma), w)(w)(\partial_x R(\sigma)) + \tilde{C}(x, R(\sigma), w)(w)(\partial_x w),$$

and we recall that for $|\xi| \leq 1/2$,

$$|C(x,\xi)| + |\partial_x C(x,\xi)| \leq \frac{K}{\operatorname{ch} x}|\xi|,$$

and

$$|\partial_\xi C(x,r)| + |\partial_x \partial_\xi C(x,\xi)| + |\partial_{\xi\xi} C(x,\xi)| \leq \frac{K}{\operatorname{ch} x},$$

so that

$$|\tilde{C}(x,u,v)| + |\partial_u \tilde{C}(x,u,v)| + |\partial_v \tilde{C}(x,u,v)| \leq \frac{K}{\operatorname{ch} x}.$$

The expected estimate of G_4 is then a straightforward consequence of these remarks.

- The last term G_5 is estimated with the same kind of arguments, using that

$$|\tilde{D}(x,u,v)| + |\partial_x \tilde{D}(x,u,v)| \leq K(|u| + |v|),$$

and that

$$|\partial_u \tilde{D}(x,u,v)| + |\partial_v \tilde{D}(x,u,v)| \leq K$$

for u and v in $B(0,1/2)$.

With these estimates, we conclude the proof of Proposition 6.1.

Now we conclude the proof of Estimate (5.26): we remark that for $s \in \mathbb{N}$, there exists C such that if $u \in H^s(\mathbb{R}^3;\mathbb{R}^3)$, then $l^i(u) \in H^s(\mathbb{R}^2;\mathbb{R})$ and

$$\|l^i(u)\|_{H^s(\mathbb{R}^2)} \leq C\|u\|_{H^s(\mathbb{R}^3)}.$$

This estimate together with Proposition 6.1 yield the expected estimates on T_1 and T_2. By difference we obtain the claimed result on T_3.

6.2 Splitting of $T_1 - T_2$

We aim to split $T_1 - T_2$ on the form : $T_1 - T_2 = \tilde{T}_a + \tilde{T}_b$, where $\tilde{T}_a$ and $\tilde{T}_b$ satisfy the following estimates: there exists K such that for all $(\sigma,\varphi,W) \in \Sigma$, if $\|(\sigma,\varphi,W)\|_{\mathcal{H}^2} \leq \gamma_1$, then

$$\|\tilde{T}_a\|_{L^1(\mathbb{R}^2)} \leq K\left(\|\nabla_Y \sigma\|_{H^2(\mathbb{R}^2)} + \|\varphi\|_{H^3(\mathbb{R}^2)} + \|W\|_{H^3(\mathbb{R}^3)}\right)^2$$

and

$$\|\tilde{T}_b\|_{L^{\frac{4}{3}}(\mathbb{R}^2)} \leq K\left(\|\nabla_Y \sigma\|_{H^2(\mathbb{R}^2)} + \|\varphi\|_{H^3(\mathbb{R}^2)} + \|W\|_{H^3(\mathbb{R}^3)}\right)\|\sigma\|_{L^4(\mathbb{R}^2)}.$$

The method is the following: each term of $T_1 - T_2$ is at least quadratic. Either it contains a product of two absorbing components (that is $\nabla_Y \sigma$, $\Delta_Y \sigma$, or φ , W and their derivatives), and we put this term in $\tilde{T}_a$, or it contains σ multiplicated by an absorbing component, and we put it in $\tilde{T}_b$ (the terms quadratic in σ are removed by using (4.14) in Section 4). Let us precise this splitting.

We recall that

$$T_1(\sigma,\varphi,W) = \gamma(\sigma)(\Delta_Y \varphi - \varphi) + \frac{\tilde{K}(\sigma)}{\tilde{g}(\sigma)}|\nabla_Y \sigma|^2 + \frac{1}{\tilde{g}(\sigma)} l^1(G),$$

$$T_2(\sigma,\varphi,W) = (1 - \tilde{g}(\sigma))\Delta_Y \sigma + \tilde{K}(\sigma)|\nabla_Y \sigma|^2 + l^2(G),$$

where $\gamma(s) = \mathcal{O}(s)$, $\tilde{g}(s) = 1 + \mathcal{O}(s)$ and $\tilde{K}(s) = \mathcal{O}(s)$.

We denote by

$$\tilde{T}_a^1 = (\frac{\tilde{K}(\sigma)}{\tilde{g}(\sigma)} - \tilde{K}(\sigma))|\nabla_Y \sigma|^2,$$

$$\tilde{T}_b^1 = \gamma(\sigma)(\Delta_Y \varphi - \varphi) - (1 - \tilde{g}(\sigma))\Delta_Y \sigma.$$

On one hand we have

$$\|\tilde{T}_a^1\|_{L^1(I\!R^2)} \leq K\|\sigma\|_{L^\infty}\|\nabla_Y \sigma\|^2_{L^2(I\!R^2)} \leq K\left(\|\nabla_Y \sigma\|_{H^2(I\!R^2)} + \|\varphi\|_{H^3(I\!R^2)} + \|W\|_{H^3(I\!R^3)}\right)^2.$$

On the other hand,

$$|\tilde{T}_b^1| \leq K|\sigma||\Delta_Y \varphi - \varphi| + K|\sigma||\Delta_Y \sigma|,$$

thus,

$$\begin{aligned}\|\tilde{T}_b^1\|_{L^{\frac{4}{3}}(I\!R^2)} \leq\ & K\|\sigma\|_{L^4(I\!R^2)}\left(\|\Delta_Y \varphi - \varphi\|_{L^2(I\!R^2)} + \|\Delta_Y \sigma\|_{L^2(I\!R^2)}\right) \\ \leq\ & K\left(\|\nabla_Y \sigma\|_{H^2(I\!R^2)} + \|\varphi\|_{H^3(I\!R^2)} + \|W\|_{H^3(I\!R^3)}\right)\|\sigma\|_{L^4(I\!R^2)}.\end{aligned}$$

Concerning the other two terms, we will split G on the form $G = G_a + G_b$ with the corresponding estimates on G_a and G_b. Let us describe this splitting for each term G_i defining G (see (4.16)).

- Concerning G_1, we recall that

  $$\Delta_Y R(\sigma) = \partial_s R(\sigma)\Delta_Y \sigma + \partial_{ss} R(\sigma|\nabla_Y \sigma|^2,$$

  and that

  $$A(r) = A(R(\sigma + w) = A(R(\sigma)) + \tilde{A}(R(\sigma, w)(w),$$

  with

  $$\tilde{A}(u, v) = \int_0^1 A'(u + sv)ds.$$

  Then we set $G_1 = G_1^a + G_1^b$ with

  $$\begin{aligned}G_1^a =\ & A(R(\sigma))(\partial_{ss}R(\sigma)|\nabla_Y \sigma|^2) + \tilde{A}(R(\sigma), w)(w)(\partial_s R(\sigma)\Delta_Y \sigma) \\ & + \tilde{A}(R(\sigma), w)(w)(\partial_{ss}R(\sigma)|\nabla_Y \sigma|^2) + 2\tilde{A}(R(\sigma), w)(w)(\Delta w), \\ G_1^b =\ & A(R(\sigma))(\partial_s R(\sigma)\Delta_Y \sigma) + A(R(\sigma))(\Delta w).\end{aligned}$$

  If (σ, φ, W) is bounded as it is assumed, we have:

  $$|G_1^a| \leq \frac{K}{\operatorname{ch} x}|\nabla_Y \sigma|^2 + \frac{K}{\operatorname{ch} x}|w||\Delta_Y \sigma| + K|w||\Delta w|,$$

  thus,

  $$\|G_1^a\|_{L^1(I\!R^3)} \leq K\left(\|\nabla_Y \sigma\|_{H^2(I\!R^2)} + \|\varphi\|_{H^3(I\!R^2)} + \|W\|_{H^3(I\!R^3)}\right)^2.$$

  On the other hand,

  $$|G_1^g| \leq \frac{K}{\operatorname{ch} x}|\sigma|(|\Delta_Y \sigma| + |\Delta w|),$$

  so

  $$\|G_1^b\|_{L^{\frac{4}{3}}(I\!R^3)} \leq K\left(\|\nabla_Y \sigma\|_{H^2(I\!R^2)} + \|\varphi\|_{H^3(I\!R^2)} + \|W\|_{H^3(I\!R^3)}\right)\|\sigma\|_{L^4(I\!R^2)}.$$

- The splitting for G_2 is the following: $G_2 = G_2^a + G_2^b$ where

  $$\begin{aligned}G_2^a =\ & B(R(\sigma))(\partial_x w, \partial_x w) + 2\tilde{B}(R(\sigma), w)(w)(\partial_x R(\sigma), \partial_x w) + \tilde{B}(R(\sigma), w)(w)(\partial_x w, \partial_x w) \\ G_2^b =\ & 2B(R(\sigma))(\partial_x R(\sigma), \partial_x w) + \tilde{B}(R(\sigma), w)(w)(\partial_x R(\sigma), \partial_x R(\sigma)).\end{aligned}$$

Since $|\partial_x R(\sigma)| \leq \dfrac{K}{\operatorname{ch} x}|\sigma|$, we have

$$|G_2^b| \leq \frac{K}{\operatorname{ch} x}|\sigma|(|\partial_x w| + |w|),$$

hence

$$\|G_2^b\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} \leq K\|w\|_{H^1(\mathbb{R}^3)}\|\sigma\|_{L^4(\mathbb{R}^2)}.$$

In addition,

$$|G_2^a| \leq K|\partial_x w|^2 + K|w||\partial_x w|,$$

so

$$\|G_2^a\|_{L^1(\mathbb{R}^3)} \leq K\|w\|^2_{H^1(\mathbb{R}^3)}.$$

- Since $\partial_i r = \partial_s R(\sigma)\partial_i \sigma + \partial_i w$ for $i = 2$ or $i = 3$, we set $G_3^a = G_3$ and $G_3^b = 0$ and we have

$$\|G_3^a\|_{L^1(\mathbb{R}^3)} \leq K(\|\nabla_Y \sigma\|^2_{L^2(\mathbb{R}^2)} + \|\nabla w\|^2_{L^2(\mathbb{R}^3)}).$$

- We define the decomposition of G_4 setting

$$\begin{aligned} G_4^a = &\quad \tilde{C}(x, R(\sigma), w)(w)(\partial_x w), \\ G_4^b = &\quad C(x, R(\sigma))(\partial_x w) + \tilde{C}(x, R(\sigma), w)(w)(\partial_x R(\sigma)). \end{aligned}$$

Since $|C(x, R(\sigma))| \leq \dfrac{K}{\operatorname{ch} x}|\sigma|$, we have

$$|G_4^b| \leq \frac{K}{\operatorname{ch} x}|\sigma|(|\partial_x w| + |w|),$$

thus

$$\|G_4^b\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} \leq K\|w\|_{H^1(\mathbb{R}^3)}\|\sigma\|_{L^4(\mathbb{R}^2)}.$$

Furthermore,

$$\|G_4^a\|_{L^1(\mathbb{R}^3)} \leq K\|w\|^2_{H^1(\mathbb{R}^3)}.$$

- Lastly, for G_5, from the Taylor expansion, we have

$$\tilde{D}(x, R(\sigma), w)(w) = \partial_\xi D(x, R(\sigma))(w) + \tilde{\tilde{D}}(x, R(\sigma), w)(w, w),$$

where

$$\tilde{\tilde{D}}(x, u, v) = \frac{1}{2}\int_0^1 (1-s)\partial_{\xi\xi} D(x, u + sv)ds.$$

We set

$$G_5^a = \tilde{\tilde{D}}(x, R(\sigma), w)(w, w) \text{ and } G_5^b = \partial_\xi D(x, R(\sigma))(w).$$

Then we have

$$|G_5^b| \leq \frac{K}{\operatorname{ch} x}|\sigma||w| \text{ so } \|G_5^b\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} \leq K\|w\|_{L^2(\mathbb{R}^3)}\|\sigma\|_{L^4(\mathbb{R}^2)},$$

and

$$\|G_5^a\|_{L^1(\mathbb{R}^3)} \leq K\|w\|^2_{L^2(\mathbb{R}^3)}.$$

Denoting $G^a = \sum_i G_i^a$ and $G^b = \sum_i G_i^b$, we have obtained that $G = G^a + G^b$ with

$$\|G^a\|_{L^1(\mathbb{R}^3)} \le K \left(\|\nabla_Y \sigma\|_{H^2(\mathbb{R}^2)} + \|\varphi\|_{H^3(\mathbb{R}^2)} + \|W\|_{H^3(\mathbb{R}^3)}\right)^2, \tag{6.34}$$

and

$$\|G^b\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} \le K \left(\|\nabla_Y \sigma\|_{H^2(\mathbb{R}^2)} + \|\varphi\|_{H^3(\mathbb{R}^2)} + \|W\|_{H^3(\mathbb{R}^3)}\right) \|\sigma\|_{L^4(\mathbb{R}^2)}. \tag{6.35}$$

We set

$$\tilde{T}_a^2 = \frac{1}{\tilde{g}(\sigma)} l^1(G^a) - l^2(G^a) \text{ and } \tilde{T}_b^2 = \frac{1}{\tilde{g}(\sigma)} l^1(G^b) - l^2(G^b).$$

By properties of the operators l^1 and l^2, (6.34) and (6.35) yield

$$\|\tilde{T}_a^2\|_{L^1(\mathbb{R}^2)} \le K \left(\|\nabla_Y \sigma\|_{H^2(\mathbb{R}^2)} + \|\varphi\|_{H^3(\mathbb{R}^2)} + \|W\|_{H^3(\mathbb{R}^3)}\right)^2,$$

and

$$\|\tilde{T}_b^2\|_{L^{\frac{4}{3}}(\mathbb{R}^2)} \le K \left(\|\nabla_Y \sigma\|_{H^2(\mathbb{R}^2)} + \|\varphi\|_{H^3(\mathbb{R}^2)} + \|W\|_{H^3(\mathbb{R}^3)}\right) \|\sigma\|_{L^4(\mathbb{R}^2)}.$$

Defining $\tilde{T}_a$ and $\tilde{T}_b$ respectively by

$$\tilde{T}_a = \tilde{T}_a^1 + \tilde{T}_a^2 \text{ and } \tilde{T}_b = \tilde{T}_b^1 + \tilde{T}_b^2,$$

we have obtained the expected decomposition. This concludes the proof of Proposition 5.1.

Acknowledgements.

It is my pleasure to thank Vincent Bruneau and Jean-François Bony of the Institut de Mathématiques de Bordeaux for their helpful suggestions. This work was supported by the ANR project SICOMAF (Simulation et Contrôle des Matériaux Ferromagnétiques).

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