Metastability of Wall Configurations in Ferromagnetic Nanowires

Gilles Carbou

LMAP - UMR CNRS 5142
Université de Pau et des Pays de l'Adour
Avenue de l'Université - BP 1155
64013 PAU CEDEX FRANCE
email: gilles.carbou@univ-pau.fr

Abstract. In this paper we consider a one dimensional model of ferromagnetic nanowire subject to a non constant electromagnetic field. Taking into account the ratio between the exchange length and the length of the wire, we prove that the wall configurations are persistent in a large time interval.

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Contents

1 Introduction

1.1 Context

The description of domain walls motion is one of the most challenging topics in ferromagnetism. In the 3-dimensional case, the pioneering works of Walker (see [36] and [41]) give a first model for planar walls in an infinite anisotropic ferromagnetic domain subject to an applied field. A first step to understand the stability of Walker's solutions with respect to the Landau-Lifschitz equation is done in

[9]. Guès and Sueur study in [22] non planar walls dynamics but only in a short time interval. In the 2-dimensional case, a wide literature in mathematics gives a few elements to describe the formation of static walls in ferromagnetic thin layers (see [1], [2], [19], [20], [34] and the references therein). The understanding of walls dynamics in one dimensional devices like ferromagnetic nanowires is very important for the applications, in particular for the storage of digital information (see [37]). For 3-dimensional models of nanowires, let us mention the works of Khün who constructs wall profiles for thin nanowires in [24] and [25]. For one dimensional models of nanowires, the description of walls motion is more complete (see [38] and the references given below). In this paper, we aim to describe the motion of several walls for the following one dimensional model of finite nanowire subject to an applied non constant magnetic field.

The wire is assimilated to the segment $[0, L]e_1$, where (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$. The magnetic moment $m = (m_1, m_2, m_3)$ is defined for $(t, x) \in \mathbb{R}^+ \times [0, L]$ and takes its values in S^2, the unit sphere of $\mathbb{R}^3$:

$$\forall\, t \geq 0,\ \forall\, x \in \mathbb{R},\ |m(t, x)| = 1 \quad \text{(saturation constraint)} \tag{1.1}$$

(we denote by $|\cdot|$ the euclidean norm in $\mathbb{R}^3$). The micromagnetism energy is given by

$$\mathcal{E}(m) = \mathcal{E}_{exch}(m) + \mathcal{E}_{dem}(m) + \mathcal{E}_{Zee}(m),$$

where the terms are defined as follows.

- The exchange energy is given by

$$\mathcal{E}_{exch}(m) = \frac{\varepsilon^2}{2} \int_{[0,L]} \left| \frac{\partial m}{\partial x} \right|^2,$$

  where we denote by ε^2 the exchange coefficient.

- The demagnetizing energy is given by

$$\mathcal{E}_{dem}(m) = \frac{1}{2} \int_{[0,L]} (|m_2|^2 + |m_3|^2).$$

  This asymptotic model for infinitely thin nanowires with round section is justified by asymptotic expansion arguments in [35] and by Γ-convergence arguments in [14]. This part of the energy forces m to be parallel to the wire axis e_1.

- The Zeeman energy describes the effects of the applied magnetic field $H_a = he_1$:

$$\mathcal{E}_{Zee}(m) = -\int_{[0,L]} hm_1.$$

The dynamics of the magnetization is described by the Landau-Lifschitz equation:

$$\frac{\partial m}{\partial t} = -m \times H_{eff} - m \times (m \times H_{eff}), \tag{1.2}$$

where the effective field H_{eff} is derived from the micromagnetism energy:

$$H_{eff} = -\nabla_m \mathcal{E} = \varepsilon^2 \partial_{xx} m - m_2 e_2 - m_3 e_3 + he_1.$$

At the boundary, as for minimizers, we assume that m satisfies the homogeneous Neumann condition:

$$\partial_x m(0) = \partial_x m(L) = 0. \tag{1.3}$$

We first remark that since it only appears $m \times H_{eff}$ in the equation, we can replace $-m_2 e_2 - m_3 e_3$ in the effective field by $m_1 e_1$. In addition, we aim to study the long time behavior of the solutions, so we perform in (1.2) the rescaling in time $\tilde{t} := \varepsilon t$. Therefore we deal with the following system:

$$\begin{cases} m : I\!R^+ \times [0,L] \to S^2, \\ \dfrac{\partial m}{\partial t} = -m \times H^\varepsilon(m,h) - m \times (m \times H^\varepsilon(m,h)), \\ H^\varepsilon(m,h) = \varepsilon \dfrac{\partial^2 m}{\partial x^2} + \dfrac{1}{\varepsilon} m_1 e_1 + \dfrac{1}{\varepsilon} h e_1, \\ \dfrac{\partial m}{\partial x}(0) = \dfrac{\partial m}{\partial x}(L) = 0. \end{cases} \tag{1.4}$$

Remark 1.1. *The time rescaling $\tilde{t} := \varepsilon t$ induces the presence of the stiff terms $\frac{1}{\varepsilon} h e_1$ and $\frac{1}{\varepsilon} m_1 e_1$ in the effective field. As we will see below, this is the good rescaling to observe walls motion with a velocity of order $\mathcal{O}(1)$ when the applied field is of order $\mathcal{O}(1)$ (see Remark 1.2 below).*

Let us assume that the applied field vanishes. The minimizers of the energy are the constant solutions $-e_1$ and $+e_1$, since the demagnetizing energy tends to align m with the wire axis. So if a magnetization configuration presents a domain in which the magnetization is almost constant, this constant will be either $-e_1$ or $+e_1$. We aim to describe configurations with $N+1$ domains (in which the magnetic moment is close to $+e_1$ or $-e_1$) separated by N walls, when the nanowire is subject to an applied field depending on x and t. Let us review the known results concerning the walls motion in one dimensional nanowires models.

For an infinite nanowire, we consider in [14] and [16] the following model. The wire is assimilated to the real line $I\!R e_1$. By rescaling in the space variable x, we remove the exchange coefficient, so that the magnetic moment m satisfies:

$$\begin{cases} m : I\!R^+ \times I\!R \to S^2, \\ \dfrac{\partial m}{\partial t} = -m \times H(m,h) - m \times (m \times H(m,h)), \\ H(m,h) = \dfrac{\partial^2 m}{\partial x^2} + m_1 e_1 + h e_1. \end{cases} \tag{1.5}$$

For this model, we consider one wall separating two domains, the left hand-side one in which the magnetization is almost equal to $-e_1$, the right hand side one in which the magnetization is almost equal to $+e_1$. Such a wall can be described by the profile M_0:

$$M_0(x) = \begin{pmatrix} \tanh x \\ 1/\cosh x \\ 0 \end{pmatrix},$$

which is an exact solution for (1.5) with $h = 0$. In addition, we remark that (1.5) is invariant by translation in the x variable and by rotation around the wire axis, so that for all $(\theta, \sigma) \in I\!R^2$, $x \mapsto \mathbf{R}_\theta M_0(x - \sigma)$ is another static solution for (1.5) with vanishing applied field describing one wall configuration, with

$$\mathbf{R}_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

We prove in [14] that M_0 is a stable static solution for (1.5) with $h = 0$.

When the applied field h is a non vanishing constant, we observe that the wall is rotating and moving: a solution for (1.5) with a constant non vanishing applied field h is given by

$$(t,x) \mapsto \mathbf{R}_{ht} M_0(x + ht).$$

We prove in [16] that this solution is stable for small values of h. Jizzini gives in [27] the threshold value of the applied field for the stability: if $|h| < 1$, then the above solution is stable. If $|h| > 1$ then this solution is linearly unstable. We remark that we have exactly the same threshold effect for constant solutions (modeling domains): $x \mapsto e_1$ is always a constant solution but it is unstable for $h \leq -1$. In the same way, $x \mapsto -e_1$ is unstable for $h \geq 1$.

Remark 1.2. *Taking into account the exchange coefficient ε^2 and the rescaling in time explained in Remark 1.1, we obtain for an infinite nanowire the model:*

$$\begin{cases} m : I\!R^+ \times I\!R \to S^2, \\ \dfrac{\partial m}{\partial t} = -m \times H^\varepsilon(m,h) - m \times (m \times H^\varepsilon(m,h)) \\ H^\varepsilon(m,h) = \varepsilon \dfrac{\partial^2 m}{\partial x^2} + \dfrac{1}{\varepsilon} m_1 e_1 + \dfrac{1}{\varepsilon} h e_1. \end{cases} \tag{1.6}$$

For a constant applied field, the motion for one wall is described by:

$$(t,x) \mapsto \mathbf{R}_{\frac{ht}{\varepsilon}} \left(M_0(\frac{x+ht}{\varepsilon}) \right),$$

so that a constant applied field of order $\mathcal{O}(1)$ induces a wall translation with a velocity of order $\mathcal{O}(1)$. Without time rescaling, the same order $\mathcal{O}(1)$ applied field would induce a wall displacement of velocity εh.

Remark 1.3. *The motion of one wall separating a left hand side $+e_1$ domain to a right hand side $-e_1$ domain is described in the model (1.6) by the profile:*

$$(t,x) \mapsto \mathbf{R}_{\frac{ht}{\varepsilon}} \left(M_0(-\frac{x-ht}{\varepsilon}) \right).$$

For this infinite wire model, we should consider solutions describing more than one wall, but the situation is very rigid: such solutions are given by periodic static solutions of the Landau-Lifschitz equation, so that the walls are periodically situated on the wire. In addition, these solutions are linearly unstable (see [32]).
For a finite nanowire, the situation is even worse. Indeed, for single wall configurations, we only obtain one solution (modulo rotation around the wire axis). In addition, this solution is centered in the middle of the wire and is unstable (see [15] for more details).
Therefore, the previous studies do not give a description for realistic configurations with several walls located in arbitrary positions on a finite wire.
In order to understand the dynamics of such configurations, we will construct approximate solutions and we will prove that the solutions of (1.4) remain close to our approximation on a large time interval when ε is small.

Our approach is inspired by the paper of Carr and Pego [17] concerning the Allen-Cahn model. They consider the scalar equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon^2 \partial_{xx} u - f(u), \\ \partial_x u(0,t) = \partial_x u(1,t) = 0, \\ u : I\!R_t^+ \times [0,1] \to I\!R, \end{cases} \tag{1.7}$$

where $f = F'$ is derived from the potential F with two non degenerate minima at the points -1 and $+1$ (the classical example is $F(u) = \frac{1}{4}(u^2-1)^2$). They prove that patterns with static transition layers are persistent on a time scale of order $\exp(C/\varepsilon)$. Roughly speaking, they consider a manifold $\mathcal{M}$ of approximate solutions with N phase transitions located at the points $h_1, h_2, \dots, h_N$. To study solutions near $\mathcal{M}$, they describe the exact solution u of (1.7) writing

$$u(t,x) = u^{h(t)}(x) + v(t,x), \tag{1.8}$$

where $u^{h(t)} \in \mathcal{M}$ is the approximate solution with N phase transitions located at the points $h_1(t), \dots, h_N(t)$, and where $v(t,\cdot)$ is orthogonal to the manifold $\mathcal{M}$ at the point $u^{h(t)}$. By analyzing the linearized problem for v, they prove that v remains small, so that the dynamics of u is essentially described by the very slow dynamics of the phase transitions patterns $u^{h(t)}$.

We adopt in our paper the same strategy. First we construct a manifold of approximate solutions. Then we describe the dynamics near this manifold in a system of coordinates in the same spirit as (1.8), and we prove that the orthogonal part (similar to v) remains small. The Landau-Lifschitz equation entails new technical difficulties compared to the Allen-Cahn equation. First, it is vectorial so that the construction of approximate solutions must be totally modified: in our case, because of the invariance of (1.4) by rotation around the wire axis, the manifold for N walls is $2N$ dimensional. In addition, the constraint $|m| = 1$ has to be taken into account for the new system of coordinates. The quasilinear character of the Landau-Lifschitz equation induces a more careful study of the non linear terms. Furthermore, contrarily to the Allen-Cahn problem for which the phase transitions are essentially static, we are able to describe the motion of the walls induced by the applied magnetic field, so that our result can be used to prove a meta-controllability for the position of the walls.

Remark 1.4. *The metastability for the Allen-Cahn equation is also obtained with energetic methods by Bronsard and Kohn [7] and more recently by Otto and Reznikoff (see [33] and the reference therein). Energetic methods are used in a vectorial framework in [5] to obtain the metastability of phase transitions. This kind of methods could be used for the Landau-Lifschitz equation without applied field.*

1.2 Statement of the main results

Let us describe our approximate solutions.

We aim to describe the behavior of a distribution of N walls separating $-e_1$ and $+e_1$ domains, subject to an applied magnetic field depending on t and x. We assume that the domain close to the boundary 0 is a $-e_1$ domain. We denote by $\sigma_1, \dots, \sigma_N$ the positions of the walls and we assume that the walls remain distant from one another and are far from the ends of the wire. We fix $\delta > 0$ such that $\delta N << L$. We assume that $(\sigma_1, \dots, \sigma_N)$ belongs to Σ_δ with:

$$\Sigma_\delta = \left\{ (\sigma_1, \dots, \sigma_N) \in I\!R^N,\ 0 < \sigma_1 - \delta < \sigma_1 + \delta < \sigma_2 - \delta < \dots < \sigma_N - \delta < \sigma_N + \delta < L \right\}.$$

For $\sigma \in \Sigma_\delta$ and $\theta = (\theta_1, \dots, \theta_N) \in I\!R^N$ we aim to describe a configuration of domains $[\sigma_i + \delta, \sigma_{i+1} - \delta]$ in which the magnetic moment is equal to $(1)^{i+1} e_1$, and these domains are separated by walls, where the wall i is centered at the point σ_i. In a central zone $[\sigma_i - 3\delta/4, \sigma_i + 3\delta/4]$ of the wall i, our profile coincides with an exact static solution of (1.4) describing one wall in an infinite wire. We use a cut off function to glue this solution to the constant ones in the neighboring domains.

By convention, we denote $\sigma_0 = -\delta$ and $\sigma_{N+1} = L + \delta$. Let us introduce a smooth cut off function $\psi : I\!R \to [0,1] \subset I\!R$ such that $\psi(s) = 0$ for $s \leq \frac{3\delta}{4}$ and $\psi(s) = 1$ for $s \geq \frac{7\delta}{8}$.
We define $\varphi_\varepsilon : I\!R \to I\!R$ by

$$\varphi_\varepsilon(x) = \frac{\pi}{2}\psi(x) - \frac{\pi}{2}\psi(-x) + (1 - \psi(x) - \psi(-x)) \arcsin\tanh\left(\frac{x}{\varepsilon}\right), \tag{1.9}$$

so that $\varphi_\varepsilon(x) = \dfrac{\pi}{2}$ for $x > \frac{7\delta}{8}$, $\varphi_\varepsilon(x) = -\dfrac{\pi}{2}$ for $x < -\frac{7\delta}{8}$ and $\varphi_\varepsilon(x) = \arcsin\tanh\left(\dfrac{x}{\varepsilon}\right)$ for $x \in [-\frac{3\delta}{4}, \frac{3\delta}{4}]$.

For a fixed (θ, σ) in $I\!R^N \times \Sigma_\delta$, we define $\varphi_\varepsilon^\sigma : [0, L] \to I\!R$ and $\mathbf{m}_\varepsilon(\theta, \sigma) : [0, L] \to S^2$ by:

- for $x \in [\sigma_{i-1}+\delta, \sigma_i - \delta]$, $i \in \{1, \dots, N+1\}$, that is for x in the i^{th} domain,

$$\varphi_\varepsilon^\sigma(x) = (-1)^i \frac{\pi}{2} \qquad \text{and} \qquad \mathbf{m}_\varepsilon(\theta,\sigma)(x) = (-1)^i e_1,$$

- for $x \in [\sigma_i - \delta, \sigma_i + \delta]$, that is for x in the i^{th} wall,

$$\varphi_\varepsilon^\sigma(x) = (-1)^{i+1}\varphi_\varepsilon(x-\sigma_i) \quad \text{and} \quad \mathbf{m}_\varepsilon(\theta,\sigma)(x) = \mathbf{R}_{\frac{\theta_i}{\varepsilon}} \begin{pmatrix} \sin\varphi_\varepsilon^\sigma(x) \\ \cos\varphi_\varepsilon^\sigma(x) \\ 0 \end{pmatrix}.$$

  In particular, in the central zone of the wall $[\sigma_i - \frac{3\delta}{4}, \sigma_i + \frac{3\delta}{4}]$, $\mathbf{m}_\varepsilon(\theta,\sigma)(x)$ coincides with $\mathbf{R}_{\frac{\theta_i}{\varepsilon}} M_0\left((-1)^{i+1}\dfrac{x-\sigma_i}{\varepsilon}\right)$, which is a static solution of (1.6) for an infinite nanowire with vanishing applied field (see Remarks 1.2 and 1.3).

Remark 1.5. *In [17], the approximate solutions u^h with $h = (h_1, \dots h_N)$ are built in a different way: between $h_i + \varepsilon$ and $h_{i+1} - \varepsilon$, u^h fits with the exact solution of $\varepsilon^2\Phi_{xx} - f(\Phi) = 0$, $\Phi(h_i) = \Phi(h_{i+1}) = 0$. In our vectorial case, this construction is not relevant since the approximate solution can take its values in different planes at the walls i and $i+1$ (the angle θ_i/ε has no reason to be equal to θ_{i+1}/ε).*

We define the set $\mathcal{M}_\delta^\varepsilon$ by

$$\mathcal{M}_\delta^\varepsilon = \left\{\mathbf{m}_\varepsilon(\theta,\sigma), \theta \in I\!R^N,\ \sigma \in \Sigma_\delta\right\}.$$

We will prove that for small values of ε and with additional assumptions on h (see below), then for initial data close to $\mathcal{M}_\delta^\varepsilon$, the solution of (1.4) remains close to the manifold $\mathcal{M}_\delta^\varepsilon$ in a large time interval. The key point of our analysis is the use of new coordinates valid in a neighborhood of $\mathcal{M}_\delta^\varepsilon$.

For $\varepsilon > 0$, $\theta \in I\!R^N$ and $\sigma \in \Sigma_\delta$, we define $\mathcal{W}_{\theta,\sigma}^\varepsilon$, the set of the $w \in H^1([0,L]; I\!R^3)$ satisfying

(i) $\forall\, x \in [0,L],\ \ w(x)\cdot \mathbf{m}_\varepsilon(\theta,\sigma)(x) = 0$,

(ii) $\forall\, i \in \{1,\dots,N\},\ \ \left\langle \partial_{\sigma_i}\mathbf{m}_\varepsilon(\theta,\sigma)|w\right\rangle = 0$,

(iii) $\forall\, i \in \{1,\dots,N\},\ \ \left\langle \partial_{\theta_i}\mathbf{m}_\varepsilon(\theta,\sigma)|w\right\rangle = 0$,

where we denote by $\cdot$ the euclidean scalar product in $I\!R^3$, and where $\langle .|.\rangle$ is the inner product in $L^2([0,L])$ (the associated $L^2([0,L])$ norm is denoted by $\|.\|_{L^2}$)

Remark 1.6. *The space $\mathcal{W}_{\theta,\sigma}^\varepsilon$ looks like the tangent space at the point $\mathbf{m}_\varepsilon(\theta,\sigma)$ of $\mathcal{M}_\delta^\varepsilon$, which is a submanifold of $H^1([0,L]; S^2)$. In particular, the pointwise orthogonality condition (i) is due to the saturation constraint $|m| = 1$ satisfied by the elements of $\mathcal{M}_\delta^\varepsilon \subset H^1([0,L]; S^2)$.*

We endow $\mathcal{W}_{\theta,\sigma}^\varepsilon$ with the norm:

$$\|w\|_\varepsilon = \left(\varepsilon\|\partial_x w\|_{L^2}^2 + \frac{1}{\varepsilon}\|w\|_{L^2}^2\right)^{\frac{1}{2}}.$$

Remark 1.7. *This norm uniformly controls the $L^\infty([0,L]$ norm (denoted by $\|.\|_{L^\infty}$), i.e. there exists a constant C such that*

$$\forall\, \varepsilon > 0, \forall\, w \in H^1([0,L]),\ \|w\|_{L^\infty} \le C\|w\|_\varepsilon.$$

We denote by $\mathrm{dist}(m, \mathcal{M}_\delta^\varepsilon)$ the distance from m to $\mathcal{M}_\delta^\varepsilon$ for the L^∞ norm:

$$\mathrm{dist}(m, \mathcal{M}_\delta^\varepsilon) = \inf_{v \in \mathcal{M}_\delta^\varepsilon} \|m - v\|_{L^\infty},$$

and we introduce the map $\nu : I\!R^3 \to I\!R$ defined by

$$\nu(\xi) = \sqrt{1 - |\xi|^2} - 1. \tag{1.10}$$

In a neighborhood of $\mathcal{M}^\varepsilon_\delta$, we use the new system of coordinates described by the following proposition, proved in Section 2.2

Proposition 1.1. *There exist $\gamma_0 > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$ and for all $m \in H^1([0,L]; S^2)$, if $dist(m, \mathcal{M}^\varepsilon_{2\delta}) \leq \gamma_0$, then there exists (θ, σ, w) with $\theta \in I\!R^N$, $\sigma \in \Sigma_\delta$ and $w \in \mathcal{W}^\varepsilon_{\theta,\sigma}$ such that*

$$m = \mathbf{m}_\varepsilon(\theta, \sigma) + w + \nu(w)\mathbf{m}_\varepsilon(\theta, \sigma).$$

In addition, for a fixed m, θ is unique in $I\!R^N/(2\pi\mathbb{Z})^N$ and (σ, w) is unique in $\Sigma_\delta \times \mathcal{W}^\varepsilon_{\theta,\sigma}$.

Our first theorem states that for a vanishing applied field, the elements of $\mathcal{M}^\varepsilon_\delta$ are metastable static quasi solutions for the problem (1.4).

Theorem 1.1. *Let $\nu_0 > 0$. There exist $\varepsilon_1 > 0$ and $\alpha_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_1[$, for all $(\bar\theta, \bar\sigma) \in I\!R^N \times \Sigma_{2\delta}$, then if $\theta_0 \in I\!R^N$, $\sigma_0 \in \Sigma_{2\delta}$ and $w_0 \in \mathcal{W}^\varepsilon_{\theta_0,\sigma_0}$ satisfy $|\theta_0 - \bar\theta| + |\sigma_0 - \bar\sigma| + \|w_0\|_\varepsilon \leq \alpha_0$, the solution m of (1.4) with vanishing applied field $h = 0$ and with initial data $m_0 = \mathbf{m}_\varepsilon(\theta_0, \sigma_0) + w_0 + \nu(w_0)\mathbf{m}_\varepsilon(\theta_0, \sigma_0)$ can be written as*

$$m(t) = \mathbf{m}_\varepsilon(\theta(t), \sigma(t)) + w(t) + \nu(w(t))\mathbf{m}_\varepsilon(\theta(t), \sigma(t)),$$

with, for all $t \in [0, e^{\frac{\delta}{4\varepsilon}}]$,

$$|\sigma(t) - \bar\sigma| + |\theta(t) - \bar\theta| + \|w(t)\|_\varepsilon \leq \nu_0.$$

Let us consider now the case of a non vanishing applied field. We assume that the wire is subject to a non constant magnetic field $(t,x) \mapsto h(t,x)e_1$. We assume that $h \in \mathcal{C}^2(I\!R^+ \times [0,L]; I\!R)$ and that there exist $\tau > 0$ and K such that:

$$\begin{aligned} &\bullet \quad \forall\,(t,x), |h(t,x)| \leq 1 - 3\tau, \\ &\bullet \quad \forall\,(t,x), \left|\frac{\partial h}{\partial x}(t,x)\right| + \left|\frac{\partial^2 h}{\partial x^2}(t,x)\right| \leq K. \end{aligned} \tag{1.11}$$

We consider $(\theta^{ref}, \sigma^{ref}) \in \mathcal{C}^1(I\!R^+; I\!R^N \times I\!R^N)$ the solution of

$$\begin{cases} \dfrac{d\sigma_i^{ref}}{dt} = (-1)^i h(t, \sigma_i^{ref}), \\[2ex] \dfrac{d\theta_i^{ref}}{dt} = h(t, \sigma_i^{ref}), \\[2ex] \sigma^{ref}(t=0) = \bar\sigma, \quad \theta^{ref}(t=0) = \bar\theta. \end{cases} \tag{1.12}$$

As we will see below, this system basically describes the motion of the walls. We assume that the applied magnetic field does not produce the collapse of the walls, *i.e.* we assume that:

$$\forall\, t > 0, \sigma^{ref}(t) \in \Sigma_{2\delta}. \tag{1.13}$$

Our second theorem states that if $(\bar\theta, \bar\sigma) \in I\!R^N \times \Sigma_{2\delta}$, then for ε small enough, if the initial data is close to $\mathbf{m}_\varepsilon(\bar\theta, \bar\sigma)$, then the solution remains close to $(t,x) \mapsto \mathbf{m}_\varepsilon(\theta^{ref}(t), \sigma^{ref}(t))$ on a time interval whose size is of order $\mathcal{O}(\frac{1}{\varepsilon})$.

Theorem 1.2. *Let $\nu_0 > 0$. There exist $\varepsilon_1 > 0$ and $\alpha_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_1[$, for all h and $(\theta^{ref}, \sigma^{ref})$ satisfying (1.11)-(1.12), then if $\theta_0 \in I\!R^N$, $\sigma_0 \in \Sigma_{2\delta}$ and $w_0 \in \mathcal{W}^{\varepsilon}_{\theta_0,\sigma_0}$ satisfy*

$$|\theta_0 - \bar{\theta}| + |\sigma_0 - \bar{\sigma}| + \|w_0\|_{\varepsilon} \leq \alpha_0,$$

then the solution m of (1.4) with initial data $m_0 = \mathbf{m}_\varepsilon(\theta_0, \sigma_0) + w_0 + \nu(w_0)\mathbf{m}_\varepsilon(\theta_0, \sigma_0)$ can be written as

$$m(t) = \mathbf{m}_\varepsilon(\theta(t), \sigma(t)) + w(t) + \nu(w(t))\mathbf{m}_\varepsilon(\theta(t), \sigma(t)),$$

with, for all $t \in [0, \frac{1}{\varepsilon}]$,

- $\|w(t)\|_\varepsilon \leq \nu_0$,
- $\sigma(t) \in \Sigma_\delta$,
- *for all i, $|\sigma_i(t) - \sigma_i^{ref}(t)| \leq \nu_0$ and $|\theta_i(t) - \theta_i^{ref}(t)| \leq \nu_0$.*

Remark 1.8. *The first assumption in (1.11) is natural to obtain metastability. Indeed, we know that the constant solution $+e_1$ (resp. $-e_1$) is unstable for the constant applied field $h = -e_1$ (resp. $+e_1$). So we should not expect stability for $|h| > 1$.*

Remark 1.9. *The dynamics described by (1.12) are deduced by the exact dynamics for one wall in the case of an infinite wire (see Remarks 1.2 and 1.3).*

We can improve this result if the applied magnetic field is constant in the walls, that is if:

$$\forall\, t \geq 0, \quad \forall\, i \in \{1, \ldots, N\}, \quad \forall\, x \in [\sigma_i^{ref}(t) - 2\delta, \sigma_i^{ref}(t) + 2\delta], \quad \partial_x h(t, x) = 0. \tag{1.14}$$

Remark 1.10. *This assumption is an hypothesis on the given applied field h.*

In this case, the solution remains close to $\mathbf{m}_\varepsilon(\theta^{ref}, \sigma^{ref})$ on a time interval of order $\mathcal{O}(e^{\frac{c}{\varepsilon}})$, as it is claimed in our third theorem.

Theorem 1.3. *Let $\nu_0 > 0$. There exist $\varepsilon_1 > 0$ and $\alpha_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_1[$, for all h and $(\theta^{ref}, \sigma^{ref})$ satisfying (1.11)-(1.12) and the additional assumption (1.14). Then if $\theta_0 \in I\!R^N$, $\sigma_0 \in \Sigma_{2\delta}$ and $w_0 \in \mathcal{W}^{\varepsilon}_{\theta_0,\sigma_0}$ satisfy $|\theta_0 - \bar{\theta}| + |\sigma_0 - \bar{\sigma}| + \|w_0\|_\varepsilon \leq \alpha_0$, the solution m of (1.4) with initial data $m_0 = \mathbf{m}_\varepsilon(\theta_0, \sigma_0) + w_0 + \nu(w_0)\mathbf{m}_\varepsilon(\theta_0, \sigma_0)$ can be written as*

$$m(t) = \mathbf{m}_\varepsilon(\theta(t), \sigma(t)) + w(t) + \nu(w(t))\mathbf{m}_\varepsilon(\theta(t), \sigma(t)),$$

with, for all $t \in [0, e^{\frac{\delta}{4\varepsilon}}]$,

- $\|w(t)\|_\varepsilon \leq \nu_0$,
- $\sigma(t) \in \Sigma_\delta$,
- *for all i, $|\sigma_i(t) - \sigma_i^{ref}(t)| \leq \nu_0$ and $|\theta_i(t) - \theta_i^{ref}(t)| \leq \nu_0$.*

Remark 1.11. *Our results do not describe neither the collapse of two walls nor when a wall goes out of the wire. Indeed Assumption (1.13) is crucial in our analysis.*

Remark 1.12. *As a corollary we should prove a quasi controllability result for the position of the walls with the applied field as command: if $\sigma^\flat$ and $\sigma^\sharp$ are in $\Sigma_{2\delta}$, there exists a command $(t, x) \mapsto h(t, x)$ such that if $\varepsilon > 0$ is small enough, if the initial data is closed to $\mathbf{m}_\varepsilon(\theta^\flat, \sigma^\flat)$ (for an arbitrary $\theta^\flat$), then after a time interval whose size is of order $\mathcal{O}(1)$, the solution is close to $\mathbf{m}_\varepsilon(\theta^\sharp, \sigma^\sharp)$ where $\theta^\sharp$ is obtained with (1.12).*

1.3 Plan of the paper

Our paper is organized as follows. In Section 2, we prove that the system of coordinates (θ, σ, w) is relevant in a neighborhood of $\mathcal{M}_\delta^\varepsilon$. The size of this neighborhood (for the L^∞ norm) does not depend on ε. We assume in Parts 3 and 4 that the applied field is constant in the walls. We relax this assumption in Part 5.
In Section 3.1 we give a detailed overview of the proof of Theorem 1.3. For the convenience of the reader, the technical points are postponed in Sections 3.2 and 3.3 (study of the linear contributions arising in the equation for w) and in Part 4 (obtention of equivalent form of the Landau-Lifschitz equation in the new variables (θ, σ, w)).
The quasi invariance of the system with respect to the rotations and the translations induces that 0 is almost an eigenvalue of order $2N$ for the linearized of the Landau-Lifschitz equation (1.4) around $\mathbf{m}_\varepsilon$. In our decomposition, this invariance is contained in the variable θ and σ so that the linearized in the variable w is coercive. Roughly speaking, for one wall, the coercivity is obtained by factorization of the operator (as in our previous papers [14], [16]). For one domain, the coercivity is straightforward. We couple these coercivity results in both cases with a trick called IMS decomposition formula using a good system of cut-off functions.
The coercivity of the linearized in the equation on w induces that w remains small for all times and that σ remains in Σ_δ in a time interval of size $\mathcal{O}(e^{\frac{\delta}{4\varepsilon}})$. Therefore the wall structure remains very stable, but a wall should go out of the wire or collapse with another wall at a very large time.
The quasi stability time for a non constant applied field is of order $\mathcal{O}(\frac{1}{\varepsilon})$ as it is proved in Section 5. We are less precise in this case because the description of the wall profiles when the wall is in a non constant magnetic field is not so precise.

Definition 1.1. *Let u be a function depending on x, ε, θ and σ with values either in $\mathbb{R}$, $\mathbb{R}^k$ or $\mathcal{L}(\mathbb{R}^2;\mathbb{R}^2)$ (linear maps from $\mathbb{R}^2$ to $\mathbb{R}^2$). We say that $u = \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$ in an interval $I \subset [0, L]$ if:*

$$\forall\, k, l, m, p \in \mathbb{N}, \exists C, \forall\, \varepsilon \in]0, 1], \forall\, \theta \in \mathbb{R}^N, \forall\, \sigma \in \Sigma_\delta, \|\frac{1}{\varepsilon^p}\partial_x^k \partial_\sigma^l \partial_\theta^m u\|_{L^\infty(I)} \leq Ce^{-\frac{\delta}{4\varepsilon}}. \tag{1.15}$$

For example, we will write that $\tanh \frac{x}{\varepsilon} = 1 + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$ for $x \geq \frac{\delta}{2}$.

Remark 1.13. *Ferromagnetism is a recent topic in mathematics. The first existence result for the Landau-Lifschitz equation is due to Visintin [40]. Existence and non uniqueness of weak solutions are proved in [3] for a simplified 3d model. Existence and asymptotic behavior studies are performed in [10]. Local in time existence of strong solutions is tackled in [11, 12, 13]. Partial regularity for particular weak solutions is obtained in [21, 23]. Numerical simulations are described in [4, 6, 28, 29, 30, 31, 39]. The main difficulties are the quasilinear character of the equations, the preservation of the saturation constraint by the numerical schemes, and the non local character of the demagnetizing field in 3d.*

1.4 Index of symbols and notations.

2 New coordinates

The goal of this section is to establish the validity of the coordinates (θ, σ, w) defined in Proposition 1.1.

2.1 Properties of the profiles $\mathbf{m}_\varepsilon(\theta, \sigma)$

We denote by h^ε the operator defined by:

$$h^\varepsilon(v) = \varepsilon \partial_{xx} v + \frac{1}{\varepsilon} v_1 e_1, \tag{2.1}$$

so that $H^\varepsilon(m, h) = h^\varepsilon(m) + \dfrac{1}{\varepsilon} h e_1$.

We aim to prove that for a fixed $(\theta, \sigma) \in \mathbb{R}^N \times \Sigma_\delta$, then $\mathbf{m}_\varepsilon(\theta, \sigma)$ is almost a static solution for (1.4) with vanishing applied field. First we prove properties concerning φ_ε defined in (1.9).

Lemma 2.1. *For all $\varepsilon > 0$,*

$$\varepsilon \frac{d^2 \varphi_\varepsilon}{dx^2} + \frac{1}{\varepsilon} \sin \varphi_\varepsilon \cos \varphi_\varepsilon = \alpha_\varepsilon \qquad \text{and} \qquad \frac{d\varphi_\varepsilon}{dx} = \frac{1}{\varepsilon} \cos \varphi_\varepsilon + \beta_\varepsilon$$

where

- $\alpha_\varepsilon = \beta_\varepsilon = 0$ *on* $]-\infty, -\frac{7\delta}{8}[\cup [-\frac{3\delta}{4}, \frac{3\delta}{4}] \cup]\frac{7\delta}{8}, +\infty[$,

- $\alpha_\varepsilon = \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$ *and* $\beta_\varepsilon = \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$ *in* $[-\frac{7\delta}{8}, -\frac{3\delta}{4}] \cup [\frac{3\delta}{4}, \frac{7\delta}{8}]$.

Proof. For $x \in]-\infty, -\frac{7\delta}{8}[$, $\varphi_\varepsilon(x) = -\frac{\pi}{2}$, and for $x \in]\frac{7\delta}{8}, +\infty[$, $\varphi_\varepsilon(x) = \frac{\pi}{2}$, so that the claimed equalities are straightforward.
On the central zone $[-\frac{3\delta}{4}, \frac{3\delta}{4}]$, $\varphi_\varepsilon(x) = \arcsin\tanh(\frac{x}{\varepsilon})$ and direct calculations yield the claimed results.
On the right hand side transitional zone $[-\frac{7\delta}{8}, -\frac{3\delta}{4}]$, $\varphi_\varepsilon(x) = -\frac{\pi}{2} + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$, and on the right hand side transitional zone $[\frac{3\delta}{4}, \frac{7\delta}{8}]$, $\varphi_\varepsilon(x) = \frac{\pi}{2} + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$, so that α_ε and β_ε are of order $\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$ on these zones.

□

Proposition 2.1. *For all $\varepsilon > 0$, for all $\theta \in I\!R^N$ and $\sigma \in \Sigma_\delta$, we have*

$$h^\varepsilon(\mathbf{m}_\varepsilon(\theta, \sigma)) = f_\varepsilon^\sigma \mathbf{m}_\varepsilon(\theta, \sigma) + \rho_\varepsilon^{\theta,\sigma},$$

where

- $f_\varepsilon^\sigma = \dfrac{1}{\varepsilon}\sin^2\varphi_\varepsilon^\sigma - \varepsilon(\partial_x\varphi_\varepsilon^\sigma)^2$,
- $\rho_\varepsilon^{\theta,\sigma} = 0$ *in the domains* $\displaystyle\bigcup_{i=0}^{N}[\sigma_i + \frac{3\delta}{4}, \sigma_{i+1} - \frac{3\delta}{4}]$,
- $\rho_\varepsilon^{\theta,\sigma} = -\mathbf{R}_{\frac{\theta_i}{\varepsilon}} \begin{pmatrix} (-1)^i\cos\varphi_\varepsilon(x-\sigma_i) \\ \sin\varphi_\varepsilon(x-\sigma_i) \\ 0 \end{pmatrix} \alpha_\varepsilon(x-\sigma_i)$ *in the wall* $[\sigma_i - \delta, \sigma_i + \delta]$

(see Lemma 2.1 for the definition of α_ε).

Proof. In the domain $[\sigma_i + \frac{3\delta}{4}, \sigma_{i+1} - \frac{3\delta}{4}]$, $\varphi_\varepsilon^\sigma(x) = (-1)^{i+1}\dfrac{\pi}{2}$ and $\mathbf{m}_\varepsilon(\theta,\sigma)(x) = (-1)^{i+1}e_i$, so

$$h^\varepsilon(\mathbf{m}_\varepsilon(\theta, \sigma)) = \frac{1}{\varepsilon}(-1)^{i+1}e_1 \quad \text{and} \quad f_\varepsilon^\sigma = \frac{1}{\varepsilon}$$

so that $\rho_\varepsilon^{\theta,\sigma} = 0$ in the domains.

In the wall $[\sigma_i - \delta, \sigma_i + \delta]$, we have

$$\mathbf{m}_\varepsilon(\theta,\sigma)(x) = \mathbf{R}_{\frac{\theta_i}{\varepsilon}} \begin{pmatrix} (-1)^{i+1}\sin\varphi_\varepsilon(x-\sigma_i) \\ \cos\varphi_\varepsilon(x-\sigma_i) \\ 0 \end{pmatrix},$$

thus by direct calculations,

$$\partial_{xx}\mathbf{m}_\varepsilon(\theta,\sigma) = -\varphi_\varepsilon''(x-\sigma_i)\mathbf{R}_{\frac{\theta_i}{\varepsilon}} \begin{pmatrix} (-1)^i\cos\varphi_\varepsilon(x-\sigma_i) \\ \sin\varphi_\varepsilon(x-\sigma_i) \\ 0 \end{pmatrix}$$
$$-(\varphi_\varepsilon'(x-\sigma_i))^2\mathbf{R}_{\frac{\theta_i}{\varepsilon}} \begin{pmatrix} (-1)^{i+1}\sin\varphi_\varepsilon(x-\sigma_i) \\ \cos\varphi_\varepsilon(x-\sigma_i) \\ 0 \end{pmatrix}.$$

In addition, $\mathbf{m}_\varepsilon(\theta,\sigma)(x) \cdot e_1 = (-1)^{i+1}\sin\varphi_\varepsilon(x-\sigma_i)$ and

$$e_1 = \mathbf{R}_{\frac{\theta_i}{\varepsilon}} \left[(-1)^{i+1}\sin\varphi_\varepsilon(x-\sigma_i)\mathbf{m}_\varepsilon(\theta,\sigma)(x) + (-1)^i\cos\varphi_\varepsilon(x-\sigma_i) \begin{pmatrix} (-1)^i\cos\varphi_\varepsilon(x-\sigma_i) \\ \sin\varphi_\varepsilon(x-\sigma_i) \\ 0 \end{pmatrix} \right].$$

Therefore,

$$h^\varepsilon(\mathbf{m}_\varepsilon(\theta,\sigma))(x) = \left[\frac{1}{\varepsilon}\sin^2\varphi_\varepsilon(x-\sigma_i) - \varepsilon(\varphi'_\varepsilon(x-\sigma_i))^2\right]\mathbf{m}_\varepsilon(\theta,\sigma)(x) + \rho_\varepsilon^{\theta,\sigma}(x)$$

where

$$\rho_\varepsilon^{\theta,\sigma}(x) = -\mathbf{R}_{\frac{\theta_i}{\varepsilon}}\begin{pmatrix}(-1)^i\cos\varphi_\varepsilon(x-\sigma_i)\\ \sin\varphi_\varepsilon(x-\sigma_i)\\ 0\end{pmatrix}\left[\varepsilon\varphi''_\varepsilon(x-\sigma_i) + \frac{1}{\varepsilon}\sin\varphi_\varepsilon(x-\sigma_i)\cos\varphi_\varepsilon(x-\sigma_i)\right].$$

Using Lemma 2.1, we conclude the proof of Proposition 2.1.

□

We describe now the properties of the derivatives of $\mathbf{m}_\varepsilon(\theta,\sigma)$ with respect to θ_i and σ_i.

Proposition 2.2. *For all $\varepsilon > 0$, for all $\theta \in I\!R^N$ and $\sigma \in \Sigma_\delta$, for all $i \in \{1,\dots,N\}$, we have*

1. *for all $x \in [0,L]$, $\mathbf{m}_\varepsilon(\theta,\sigma)(x)\cdot\partial_{\sigma_i}\mathbf{m}_\varepsilon(\theta,\sigma)(x) = \mathbf{m}_\varepsilon(\theta,\sigma)(x)\cdot\partial_{\theta_i}\mathbf{m}_\varepsilon(\theta,\sigma)(x) = 0$,*
2. *for all $x \in [0,L]$, $\partial_{\sigma_i}\mathbf{m}_\varepsilon(\theta,\sigma)(x)\cdot\partial_{\theta_i}\mathbf{m}_\varepsilon(\theta,\sigma)(x) = 0$,*
3. *$\|\partial_{\theta_i}\mathbf{m}_\varepsilon\|_{L^1} + \|\partial_{\sigma_i}\mathbf{m}_\varepsilon\|_{L^1} \le K$, where K does not depend on $\varepsilon > 0$, $\theta \in I\!R^N$ and $\sigma \in \Sigma_\delta$,*
4. *$\langle\partial_{\theta_i}\mathbf{m}_\varepsilon|\partial_{\theta_i}\mathbf{m}_\varepsilon\rangle = \frac{2}{\varepsilon} + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$,*
5. *$\langle\partial_{\sigma_i}\mathbf{m}_\varepsilon|\partial_{\sigma_i}\mathbf{m}_\varepsilon\rangle = \frac{2}{\varepsilon} + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$.*

Proof. For $x \in [0,L]\setminus[\sigma_i - \frac{7\delta}{8}, \sigma_i + \frac{7\delta}{8}]$, $\mathbf{m}_\varepsilon(\theta,\sigma)(x)$ does not depend on θ_i and σ_i, so that

$$\forall\, x \in [0,L]\setminus[\sigma_i-\delta,\sigma_i+\delta], \quad \partial_{\sigma_i}\mathbf{m}_\varepsilon(\theta,\sigma)(x) = \partial_{\theta_i}\mathbf{m}_\varepsilon(\theta,\sigma)(x) = 0. \tag{2.2}$$

For $x \in [\sigma_i-\delta,\sigma_i+\delta]$,

$$\mathbf{m}_\varepsilon(\theta,\sigma)(x) = \mathbf{R}_{\frac{\theta_i}{\varepsilon}}\begin{pmatrix}(-1)^{i+1}\sin\varphi_\varepsilon(x-\sigma_i)\\ \cos\varphi_\varepsilon(x-\sigma_i)\\ 0\end{pmatrix}, \tag{2.3}$$

so

$$\partial_{\theta_i}\mathbf{m}_\varepsilon(\theta,\sigma)(x) = \frac{1}{\varepsilon}\mathbf{R}'_{\frac{\theta_i}{\varepsilon}}\begin{pmatrix}(-1)^{i+1}\sin\varphi_\varepsilon(x-\sigma_i)\\ \cos\varphi_\varepsilon(x-\sigma_i)\\ 0\end{pmatrix},$$

where

$$\mathbf{R}'_\theta = \begin{pmatrix}0 & 0 & 0\\ 0 & -\sin\theta & -\cos\theta\\ 0 & \cos\theta & -\sin\theta\end{pmatrix}.$$

We remark that

$$\mathbf{R}'_\theta = \mathbf{R}_\theta\mathbf{R}_{-\theta}\mathbf{R}'_\theta = \mathbf{R}_\theta\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix},$$

so

$$\partial_{\theta_i}\mathbf{m}_\varepsilon(\theta,\sigma)(x) = \frac{\cos\varphi_\varepsilon(x-\sigma_i)}{\varepsilon}\mathbf{R}_{\frac{\theta_i}{\varepsilon}}\begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{2.4}$$

On the other hand,

$$\partial_{\sigma_i}\mathbf{m}_\varepsilon(\theta,\sigma)(x) = \varphi'_\varepsilon(x-\sigma_i)\mathbf{R}_{\frac{\theta_i}{\varepsilon}}\begin{pmatrix}(-1)^i\cos\varphi_\varepsilon(x-\sigma_i)\\ \sin\varphi_\varepsilon(x-\sigma_i)\\ 0\end{pmatrix}. \tag{2.5}$$

The point wise orthogonality conditions 1 and 2 are direct consequences of (2.3), (2.4) and (2.5). Concerning the L^1 estimate, we have:

$$\|\partial_{\sigma_i}\mathbf{m}_\varepsilon\|_{L^1} = \int_{-\delta}^{\delta} \varphi_\varepsilon' = \pi.$$

From Lemma 2.1 and Equation (2.4), we have:

$$\|\partial_{\theta_i}\mathbf{m}_\varepsilon\|_{L^1} = \|\partial_{\sigma_i}\mathbf{m}_\varepsilon\|_{L^1} + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}) = \pi + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}).$$

In addition, we have

$$\begin{aligned}
\left\langle \partial_{\sigma_i}\mathbf{m}_\varepsilon | \partial_{\sigma_i}\mathbf{m}_\varepsilon \right\rangle &= \int_{\sigma_i-\delta}^{\sigma_i+\delta} |\varphi_\varepsilon'|^2 dx \\
&= \int_{\sigma_i-\delta/2}^{\sigma_i+\delta/2} |\varphi_\varepsilon'|^2 dx + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}) \\
&= \int_{\sigma_i-\delta/2}^{\sigma_i+\delta/2} \frac{dx}{|\varepsilon \cosh \frac{x-\sigma_i}{\varepsilon}|^2} + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}) \\
&= \frac{1}{\varepsilon}\int_{-\frac{\delta}{2\varepsilon}}^{\frac{\delta}{2\varepsilon}} \frac{dy}{\cosh^2 y} + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}) \\
&= \frac{2}{\varepsilon} + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}).
\end{aligned}$$

The same holds for $\left\langle \partial_{\theta_i}\mathbf{m}_\varepsilon | \partial_{\theta_i}\mathbf{m}_\varepsilon \right\rangle$.

□

Remark 2.1. *It is clear that for $i \neq j$, then $\left\langle \partial_{\theta_i}\mathbf{m}_\varepsilon | \partial_{\theta_j}\mathbf{m}_\varepsilon \right\rangle = \left\langle \partial_{\sigma_i}\mathbf{m}_\varepsilon | \partial_{\sigma_j}\mathbf{m}_\varepsilon \right\rangle = 0$. In addition $\left\langle \partial_{\theta_i}\mathbf{m}_\varepsilon | \partial_{\sigma_j}\mathbf{m}_\varepsilon \right\rangle = 0$ for all (i,j).*

2.2 Inversion Theorem for the new coordinates

We aim to use the coordinates (θ, σ, w) in a neighborhood of the manifold $\mathcal{M}^\varepsilon_{2\delta}$.

For $u \in L^\infty([0,L]; I\!R^3)$ and $v = (\theta, \sigma) \in I\!R^N \times \Sigma_\delta$, we define $\mathcal{F}(u,v) \in I\!R^{2N}$ by

$$\mathcal{F}(u,v) = \begin{pmatrix} \left\langle u | \partial_{\theta_1}\mathbf{m}_\varepsilon(v) \right\rangle \\ \cdot \\ \cdot \\ \left\langle u | \partial_{\theta_N}\mathbf{m}_\varepsilon(v) \right\rangle \\ \left\langle u | \partial_{\sigma_1}\mathbf{m}_\varepsilon(v) \right\rangle \\ \cdot \\ \cdot \\ \left\langle u | \partial_{\sigma_N}\mathbf{m}_\varepsilon(v) \right\rangle \end{pmatrix}.$$

If we can write $m = \mathbf{m}_\varepsilon(\theta,\sigma) + w + \nu(w)\mathbf{m}_\varepsilon(\theta,\sigma)$, taking the L^2 inner product with $\partial_{\theta_i}\mathbf{m}_\varepsilon(\theta,\sigma)$ and with $\partial_{\sigma_i}\mathbf{m}_\varepsilon(\theta,\sigma)$, by the orthogonality conditions on w, and with *1.* in Proposition 2.2, we obtain that:

$$\mathcal{F}(m, (\theta,\sigma)) = 0.$$

We aim to prove the following lemma.

Lemma 2.2. *There exists $\nu_0 > 0$ such that for all $\varepsilon \in]0,\delta]$, for all $(\bar\theta,\bar\sigma) \in I\!R^N \times \Sigma_{2\delta}$, for all $u \in L^\infty([0,L])$, if $\|u - \mathbf{m}_\varepsilon(\bar\theta,\bar\sigma)\|_{L^\infty} \le \nu_0$ then there exists one and only one pair $(\theta,\sigma) \in I\!R^N \times \Sigma_\delta$ in a neighborhood of $(\bar\theta,\bar\sigma)$ such that*

$$\mathcal{F}(u,(\theta,\sigma)) = 0.$$

Proof. As a classical inversion theorem, the proof is based on a fixed point theorem. We must check that the size of the neighborhood of $\mathcal{M}^\varepsilon_{2\delta}$ can be chosen independently of $\varepsilon > 0$ small enough.

We have $\mathcal{F}(\mathbf{m}_\varepsilon(\theta,\sigma),(\theta,\sigma)) = 0$.

In addition, from (2.2), (2.4) and (2.5), we obtain that:

- if $i \neq j$, $\partial_{\sigma_i\sigma_j}\mathbf{m}_\varepsilon = \partial_{\sigma_i\theta_j}\mathbf{m}_\varepsilon = \partial_{\theta_i\theta_j}\mathbf{m}_\varepsilon = 0$,
- on $[0,L]\setminus[\sigma_i-\delta,\sigma_i+\delta]$, $\partial_{\sigma_i\sigma_i}\mathbf{m}_\varepsilon = \partial_{\sigma_i\theta_i}\mathbf{m}_\varepsilon = \partial_{\theta_i\theta_i}\mathbf{m}_\varepsilon = 0$,
- for $x \in [\sigma_i-\delta,\sigma_i+\delta]$,

$$\begin{aligned}
\partial_{\sigma_i\sigma_i}\mathbf{m}_\varepsilon(\theta,\sigma)(x) = \quad & -\varphi_\varepsilon''(x-\sigma_i)\mathbf{R}_{\frac{\theta_i}{\varepsilon}}\begin{pmatrix}(-1)^i\cos\varphi_\varepsilon(x-\sigma_i)\\ \sin\varphi_\varepsilon(x-\sigma_i)\\ 0\end{pmatrix}\\
& +(\varphi_\varepsilon'(x-\sigma_i))^2\mathbf{R}_{\frac{\theta_i}{\varepsilon}}\begin{pmatrix}(-1)^i\sin\varphi_\varepsilon(x-\sigma_i)\\ -\cos\varphi_\varepsilon(x-\sigma_i)\\ 0\end{pmatrix}\\
= \quad & -\left(\varphi_\varepsilon'(x-\sigma_i)\right)^2\mathbf{m}_\varepsilon(\theta,\sigma)(x) + \frac{\sin\varphi_\varepsilon(x-\sigma_i)}{\varepsilon}\partial_{\sigma_i}\mathbf{m}_\varepsilon(\theta,\sigma)(x)\\
& -\beta_\varepsilon(x-\sigma_i)\frac{\sin\varphi_\varepsilon(x-\sigma_i)}{\varepsilon}\mathbf{R}_{\frac{\theta_i}{\varepsilon}}\begin{pmatrix}(-1)^i\sin\varphi_\varepsilon(x-\sigma_i)\\ -\cos\varphi_\varepsilon(x-\sigma_i)\\ 0\end{pmatrix},
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
\partial_{\sigma_i\theta_i}\mathbf{m}_\varepsilon(\theta,\sigma)(x) = \quad & \varphi_\varepsilon'(x-\sigma_i)\frac{\sin(\varphi_\varepsilon(x-\sigma_i))}{\varepsilon}\mathbf{R}_{\frac{\theta_i}{\varepsilon}}\begin{pmatrix}0\\0\\1\end{pmatrix}\\
= \quad & \frac{\sin\varphi_\varepsilon(x-\sigma_i)}{\varepsilon}\partial_{\theta_i}\mathbf{m}_\varepsilon(\theta,\sigma)(x) - \beta_\varepsilon(x-\sigma_i)\frac{\sin\varphi_\varepsilon(x-\sigma_i)}{\varepsilon}\mathbf{R}_{\frac{\theta_i}{\varepsilon}}\begin{pmatrix}0\\0\\1\end{pmatrix},\\
\partial_{\theta_i\theta_i}\mathbf{m}_\varepsilon(\theta,\sigma)(x) = \quad & \frac{\cos\varphi_\varepsilon(x-\sigma_i)}{\varepsilon^2}\mathbf{R}_{\frac{\theta_i}{\varepsilon}}\begin{pmatrix}0\\-1\\0\end{pmatrix}\\
= \quad & -\frac{\cos^2\varphi_\varepsilon(x-\sigma_i)}{\varepsilon^2}\mathbf{m}_\varepsilon(\theta,\sigma)(x) - \frac{\sin\varphi_\varepsilon(x-\sigma_i)}{\varepsilon}\partial_{\sigma_i}\mathbf{m}_\varepsilon(\theta,\sigma)(x)\\
& +\frac{\sin\varphi_\varepsilon(x-\sigma_i)}{\varepsilon}\beta_\varepsilon(x-\sigma_i)\mathbf{R}_{\frac{\theta_i}{\varepsilon}}\begin{pmatrix}0\\-1\\0\end{pmatrix}.
\end{aligned} \tag{2.7}$$

Therefore, for $x \in [\sigma_i - \delta, \sigma_i + \delta]$,

$$\mathbf{m}_\varepsilon(\theta,\sigma)(x) \cdot \partial_{\sigma_i\sigma_i}\mathbf{m}_\varepsilon(\theta,\sigma)(x) = -(\varphi_\varepsilon'(x-\sigma_i))^2$$

$$\mathbf{m}_\varepsilon(\theta,\sigma)(x) \cdot \partial_{\sigma_i\sigma_i}\mathbf{m}_\varepsilon(\theta,\sigma)(x) = 0$$

$$\mathbf{m}_\varepsilon(\theta,\sigma)(x) \cdot \partial_{\sigma_i\sigma_i}\mathbf{m}_\varepsilon(\theta,\sigma)(x) = -\frac{\cos^2\varphi_\varepsilon(x-\sigma_i)}{\varepsilon^2}$$

so that, using Proposition 2.1, we obtain that

$$\partial_v\mathcal{F}(\mathbf{m}_\varepsilon(\theta,\sigma),(\theta,\sigma)) = \frac{2}{\varepsilon}\begin{pmatrix} -I_N & 0_N \\ 0_N & I_N \end{pmatrix} + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}). \tag{2.8}$$

We define $\mathcal{G} : L^\infty([0,L]) \times I\!R^N \times \Sigma_\delta \to I\!R^{2N}$ by

$$\mathcal{G}(u,v) = v - \frac{\varepsilon}{2}\begin{pmatrix} -I_N & 0_N \\ 0_N & I_N \end{pmatrix}\mathcal{F}(u,v).$$

We aim to prove that there exists $\nu_0 > 0$ independent of ε such that for $(\bar\theta,\bar\sigma)$ fixed in $I\!R^N \times \Sigma_{2\delta}$, for $u \in L^\infty([0,L])$, if $\|u - \mathbf{m}_\varepsilon(\bar\theta,\bar\sigma)\|_{L^\infty} \le \nu_0$, then $v \mapsto \mathcal{G}(u,v)$ admits one and only one fixed point in a neighborhood of $(\bar\theta,\bar\sigma)$.
We have:

$$\begin{aligned}\mathcal{G}(u,v_2) - \mathcal{G}(u,v_1) = \quad & v_2 - v_1 - \frac{\varepsilon}{2}\begin{pmatrix} -I_N & 0_N \\ 0_N & I_N \end{pmatrix}\int_0^1 \partial_v\mathcal{F}(u, v_1 + s(v_2-v_1))(v_2-v_1)ds \\ = \quad & -\frac{\varepsilon}{2}\begin{pmatrix} -I_N & 0_N \\ 0_N & I_N \end{pmatrix}\int_0^1 \left(\partial_v\mathcal{F}(u,v_1+s(v_2-v_1)) - \partial_v\mathcal{F}(\bar u,\bar v)\right)(v_2-v_1)ds \\ & + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})(v_2-v_1)\end{aligned}$$

where $\bar v = (\bar\theta,\bar\sigma)$ and $\bar u = \mathbf{m}_\varepsilon(\bar v)$, using (2.8). We have now:

$$\begin{aligned}\left|\partial_v\mathcal{F}(u,v_1+s(v_2-v_1)) - \partial_v\mathcal{F}(\bar u,\bar v)\right| \le \quad & \left|\partial_v\mathcal{F}(u,v_1+s(v_2-v_1)) - \partial_v\mathcal{F}(\bar u, v_1+s(v_2-v_1))\right| \\ & + \left|\partial_v\mathcal{F}(\bar u, v_1+s(v_2-v_1)) - \partial_v\mathcal{F}(\bar u,\bar v)\right|.\end{aligned}$$

On the one hand,

$$\begin{aligned}&\left|\partial_v\mathcal{F}(u,v_1+s(v_2-v_1)) - \partial_v\mathcal{F}(\bar u, v_1+s(v_2-v_1))\right| \le \\ &\qquad K\|u-\bar u\|_{L^\infty}\max_i\left(\|\partial_{\theta_i\theta_i}\mathbf{m}_\varepsilon\|_{L^2} + \|\partial_{\theta_i\sigma_i}\mathbf{m}_\varepsilon\|_{L^2} + \|\partial_{\sigma_i\sigma_i}\mathbf{m}_\varepsilon\|_{L^2}\right) \\ &\qquad \le \frac{K}{\varepsilon}\|u-\bar u\|_{L^\infty}\end{aligned}$$

On the other hand,

$$\left|\partial_v\mathcal{F}(\bar u, v_1+s(v_2-v_1)) - \partial_v\mathcal{F}(\bar u,\bar v)\right| \le \max_{v\in[\bar v, v_1+s(v_2-v_1)]}\left|\partial_{vv}\mathcal{F}(\bar u,v)\right|\left|v_1+s(v_2-v_1)-\bar v\right|.$$

The second derivative of $\mathcal{F}$ can be estimated as follows:

$$\left|\partial_{vv}\mathcal{F}(\bar u,v)\right| \le K\max_i\left(\|\partial_{\sigma_i\sigma_i\sigma_i}\mathbf{m}_\varepsilon\|_{L^1} + \|\partial_{\sigma_i\sigma_i\theta_i}\mathbf{m}_\varepsilon\|_{L^1} + \|\partial_{\sigma_i\theta_i\theta_i}\mathbf{m}_\varepsilon\|_{L^1} + \|\partial_{\theta_i\theta_i\theta_i}\mathbf{m}_\varepsilon\|_{L^1}\right).$$

We have

- $\partial_{\theta_i\theta_i\theta_i}\mathbf{m}_\varepsilon = -\frac{1}{\varepsilon^2}\partial_{\theta_i}\mathbf{m}_\varepsilon + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$ so

$$\|\partial_{\theta_i\theta_i\theta_i}\mathbf{m}_\varepsilon\|_{L^1} \leq \frac{K}{\varepsilon^2},$$

- $\partial_{\sigma_i\theta_i\theta_i}\mathbf{m}_\varepsilon = (-1)^{i+1}\frac{\sin\varphi_\varepsilon^\sigma}{\varepsilon}\left(-\frac{\cos^2\varphi_\varepsilon^\sigma}{\varepsilon^2}\mathbf{m}_\varepsilon + (-1)^i\frac{\sin\varphi_\varepsilon^\sigma}{\varepsilon}\partial_{\sigma_i}\mathbf{m}_\varepsilon + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})\right)$, so

$$\|\partial_{\sigma_i\theta_i\theta_i}\mathbf{m}_\varepsilon\|_{L^1} \leq \frac{K}{\varepsilon^2},$$

- $\partial_{\sigma_i\sigma_i\theta_i}\mathbf{m}_\varepsilon = -\frac{1}{\varepsilon^2}\partial_{\theta_i}\mathbf{m}_\varepsilon + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$ so

$$\|\partial_{\sigma_i\sigma_i\theta_i}\mathbf{m}_\varepsilon\|_{L^1} \leq \frac{K}{\varepsilon^2},$$

- $\partial_{\sigma_i\sigma_i\theta_i}\mathbf{m}_\varepsilon = \frac{3}{\varepsilon^2}\sin\varphi_\varepsilon^\sigma\cos\varphi_\varepsilon^\sigma\mathbf{m}_\varepsilon + \frac{\sin^2\varphi_\varepsilon^\sigma}{\varepsilon^2}\partial_{\sigma_i}\mathbf{m}_\varepsilon + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$ so

$$\|\partial_{\sigma_i\sigma_i\sigma_i}\mathbf{m}_\varepsilon\|_{L^1} \leq \frac{K}{\varepsilon^2}.$$

So

$$\left|\partial_v\mathcal{F}(\bar{u}, v_1 + s(v_2 - v_1)) - \partial_v\mathcal{F}(\bar{u}, \bar{v})\right| \leq \frac{K}{\varepsilon^2}(|v_1 - \bar{v}| + |v_2 - \bar{v}|).$$

Therefore, there exists K independent of u, $\bar{u}$, v_1, v_2 and $\bar{v}$ such that

$$\left|\mathcal{G}(u, v_1) - \mathcal{G}(u, v_2)\right| \leq K\left(\|u - \bar{u}\|_{L^\infty} + \frac{1}{\varepsilon}|v_1 - \bar{v}| + \frac{1}{\varepsilon}|v_2 - \bar{v}|\right)|v_2 - v_1|. \tag{2.9}$$

In addition,

$$\left|\mathcal{G}(u, v) - \mathcal{G}(\bar{u}, \bar{v})\right| \leq \left|\mathcal{G}(u, v) - \mathcal{G}(\bar{u}, v)\right| + \left|\mathcal{G}(\bar{u}, v) - \mathcal{G}(\bar{u}, \bar{v})\right|.$$

We have

- $$\begin{aligned}\left|\mathcal{G}(u, v) - \mathcal{G}(\bar{u}, v)\right| &\leq K\varepsilon\left|\mathcal{F}(u, v) - \mathcal{F}(\bar{u}, v)\right| \\ &\leq K\varepsilon\|u - \bar{u}\|_{L^\infty}\max_i(\|\partial_{\theta_i}\mathbf{m}_\varepsilon\|_{L^1} + \|\partial_{\sigma_i}\mathbf{m}_\varepsilon\|_{L^1}) \\ &\leq K\|u - \bar{u}\|_{L^\infty}\varepsilon.\end{aligned}$$
- $\left|\mathcal{G}(\bar{u}, v) - \mathcal{G}(\bar{u}, \bar{v})\right| \leq K\left(\|u - \bar{u}\|_{L^\infty} + \frac{1}{\varepsilon}|v - \bar{v}|\right)|v - \bar{v}|.$

So we have obtained that there exists a universal constant K such that:

$$|\mathcal{G}(u, v) - \bar{v}| \leq K\varepsilon\|u - \bar{u}\|_{L^\infty} + K\left(\|u - \bar{u}\|_{L^\infty} + \frac{1}{\varepsilon}|v - \bar{v}|\right)|v - \bar{v}|$$

and

$$|\mathcal{G}(u, v_1) - \mathcal{G}(u, v_2)| \leq K\left(\|u - \bar{u}\|_{L^\infty} + \frac{1}{\varepsilon}|v_1 - \bar{v}| + \frac{1}{\varepsilon}|v_2 - \bar{v}|\right)|v_2 - v_1|.$$

We fix $\alpha_1 > 0$ and $\alpha_2 > 0$ such that $K(\alpha_1 + \alpha_2) \leq \frac{1}{4}$, $K\alpha_2 \leq \frac{1}{4}$ and $K\alpha_1 \leq \frac{\alpha_2}{2}$.

If $\|u-\bar{u}\|_{L^\infty} \leq \alpha_1$, if $|v-\bar{v}| \leq \alpha_2\varepsilon$, then

$$|\mathcal{G}(u,v)-\bar{u}| \leq K\varepsilon\alpha_1 + K(\alpha_1+\alpha_2)|v-\bar{v}| \leq \frac{3}{4}\alpha_2\varepsilon,$$

so $v \mapsto \mathcal{G}(u,v)$ maps $B(\bar{v},\alpha_2\varepsilon)$ into itself.

In addition, $K(\alpha_1+2\alpha_2) \leq \dfrac{1}{2}$ so this map is contracting, so the fixed point exists and is unique. Taking $\nu_0 = \alpha_1$ we conclude the proof of Lemma 2.2.

□

Proof of Proposition 1.1.

Let $m \in H^1([0,L];S^2)$ such that $\operatorname{dist}(m,\mathcal{M}^\varepsilon_{2\delta}) \leq \nu_0$. So there exists $\bar{v} = (\bar{\theta},\bar{\sigma})$ such that $\|m - \mathbf{m}_\varepsilon(\bar{\theta},\bar{\sigma})\|_{L^\infty([0,L])} \leq \nu_0$.
From Lemma 2.2, we find $v = (\theta,\sigma) \in B(\bar{v},\alpha_2\varepsilon)$ such that $\mathcal{F}(m,v) = 0$. We define w by $w = -\mathbf{m}_\varepsilon(v) \times (\mathbf{m}_\varepsilon(v) \times m)$ so that we have by construction:

$$m = \mathbf{m}_\varepsilon(v) + w + \nu(w)\mathbf{m}_\varepsilon,$$

with $m \cdot \mathbf{m}_\varepsilon(v) = 0$, and $\mathcal{F}(m,v) = \mathcal{F}(w,v) = 0$. In addition, w has the same regularity as m. This concludes the proof of Proposition 1.1. □

3 Meta-stability for a piecewise constant applied field

We remark that Theorem 1.1 is a direct consequence of Theorem 1.3 so that we only discuss this last theorem. In Section 3.1, we give an overview of the proof of Theorem 1.3. We detail the technical parts in Sections 3.2 and 3.3, and in Section 4 for the longest calculations.

3.1 Sketch of the proof of Theorem 1.3

We consider h and $(\theta^{ref},\sigma^{ref})$ satisfying (1.11)-(1.12)-(1.14), and we denote by h_i the value of the magnetic applied field in the wall i:

$$\forall\, i \in \{1,\dots,N\},\ \forall\, t,\ h_i(t) = h(t,\sigma_i^{ref}(t)). \tag{3.10}$$

The proof of Theorem 1.3 is divided into the following five steps.

First step: equivalent system in the new coordinates

While the solution $m(t,x)$ with initial data $\mathbf{m}_\varepsilon(\bar{\theta},\bar{\sigma})$ remains in a neighborhood of $\mathcal{M}^\varepsilon_\delta$, using Proposition 1.1, we write it on the form:

$$m(t,x) = \mathbf{m}_\varepsilon(\theta(t),\sigma(t)) + w(t) + \nu(w(t))\mathbf{m}_\varepsilon(\theta(t),\sigma(t)), \tag{3.11}$$

where $\theta \in \mathcal{C}^1(0,T;I\!R^N)$, $\sigma \in \mathcal{C}^1(0,T;\Sigma_\delta)$ and for all t, $w(t) \in \mathcal{W}^\varepsilon_{\theta(t),\sigma(t)}$.

By plugging (3.11) in the Landau-Lifschitz system (1.4), by taking the inner product with $\partial_{\theta_i}\mathbf{m}_\varepsilon$ and $\partial_{\sigma_i}\mathbf{m}_\varepsilon$, we prove the following proposition.

Proposition 3.1. *While $m = \mathbf{m}_\varepsilon(\theta(t),\sigma(t)) + w(t) + \nu(w(t))\mathbf{m}_\varepsilon(\theta(t),\sigma(t))$ satisfies $|\sigma(t)-\sigma^{ref}(t)| \leq \delta$, then m satisfies the Landau-Lifschitz equation (1.4) if and only if $(\theta(t),\sigma(t),w(t))$ satisfies the following system* (3.12)-(3.13)-(3.14)*:*

$$\frac{d\theta_i}{dt} = h_i + a_\varepsilon^{\theta_i} - h_i\Pi_\varepsilon^{\theta_i}(w) + l_\varepsilon^{\theta_i}(w) + G_\varepsilon^{\theta_i}(\theta_i,\sigma_i,w), \tag{3.12}$$

$$\frac{d\sigma_i}{dt} = (-1)^i h_i + a_\varepsilon^{\sigma_i} - h_i\Pi_\varepsilon^{\sigma_i}(w) + l_\varepsilon^{\sigma_i}(w) + G_\varepsilon^{\sigma_i}(\theta_i,\sigma_i,w), \tag{3.13}$$

$$\frac{\partial w}{\partial t} = a_\varepsilon + \Lambda_\varepsilon w + P_\varepsilon w + l^\varepsilon w + G^\varepsilon(w,\theta,\sigma), \tag{3.14}$$

where

- $a_\varepsilon^{\theta_i} : \mathbb{R}^+ \times [0,L] \to \mathbb{R}$, $a_\varepsilon^{\sigma_i} : \mathbb{R}^+ \times [0,L] \to \mathbb{R}$, $l_\varepsilon^{\theta_i} : \mathbb{R}^+ \times [0,L] \to \mathcal{L}(\mathbb{R}^3;\mathbb{R})$ *and* $l_\varepsilon^{\sigma_i} : \mathbb{R}^+ \times [0,L] \to \mathcal{L}(\mathbb{R}^3;\mathbb{R})$ *are of order* $\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$,
- $a_\varepsilon : \mathbb{R}^+ \times [0,L] \to \mathbb{R}$ *and* $l^\varepsilon : \mathbb{R}^+ \times [0,L] \to \mathcal{L}(\mathcal{W}^\varepsilon_{\theta,\sigma};\mathbb{R}^3)$ *are of order* $\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$,
- h_i *is defined by (3.10),*
- *the linear parts* $\Pi_\varepsilon^{\theta_i}(w)$ *and* $\Pi_\varepsilon^{\sigma_i}(w)$ *are given by*

$$\begin{aligned}
\Pi_\varepsilon^{\theta_i}(w) &= \left\langle \sin\varphi_\varepsilon^\sigma \partial_{\theta_i}\mathbf{m}_\varepsilon | w \right\rangle = \int_{\sigma_i-\delta}^{\sigma_i+\delta} \sin\varphi_\varepsilon^\sigma w \cdot \partial_{\theta_i}\mathbf{m}_\varepsilon, \\
\Pi_\varepsilon^{\sigma_i}(w) &= \left\langle \sin\varphi_\varepsilon^\sigma \partial_{\sigma_i}\mathbf{m}_\varepsilon | w \right\rangle = \int_{\sigma_i-\delta}^{\sigma_i+\delta} \sin\varphi_\varepsilon^\sigma w \cdot \partial_{\sigma_i}\mathbf{m}_\varepsilon,
\end{aligned} \tag{3.15}$$

  and satisfy that there exists a constant K *such that*

$$|\Pi_\varepsilon^{\theta_i}(w)| + |\Pi_\varepsilon^{\sigma_i}(w)| \leq K\|w\|_\varepsilon, \tag{3.16}$$

- *the linear operator* Λ_ε *is given by:*

$$\Lambda_\varepsilon w = -\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w) - \mathbf{m}_\varepsilon \times (\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)),$$

  with

$$\mathcal{H}^\varepsilon(w) = -h^\varepsilon(w) + f_\varepsilon^\sigma w = -\varepsilon\partial_{xx}w - \frac{1}{\varepsilon}w_1 e_1 + f_\varepsilon^\sigma w, \tag{3.17}$$

- *the linear perturbation due to the applied magnetic field writes*

$$P_\varepsilon w = -\frac{h}{\varepsilon} w \times e_1 - \frac{h}{\varepsilon} w_1 \mathbf{m}_\varepsilon - \frac{h}{\varepsilon}\sin\varphi_\varepsilon^\sigma w + h\varepsilon\Pi_\varepsilon^{\theta_i}(w)\partial_{\theta_i}\mathbf{m}_\varepsilon + h\varepsilon\Pi_\varepsilon^{\sigma_i}(w)\partial_{\sigma_i}\mathbf{m}_\varepsilon,$$

- *the non linear term* $G^\varepsilon(w,\theta,\sigma)$ *satisfies*

$$\|G^\varepsilon(w,\theta,\sigma)\|_{L^2} \leq K\|w\|_\varepsilon \left(\|\varepsilon\partial_{xx}w\|_{L^2} + \frac{1}{\varepsilon}\|w\|_{L^2} \right),$$

- *the non linear terms* $G_\varepsilon^{\theta_i}$ *and* $G_\varepsilon^{\sigma_i}$ *can be estimated as follows:*

$$|G_\varepsilon^{\theta_i}(\theta_i,\sigma_i,w)| + |G_\varepsilon^{\sigma_i}(\theta_i,\sigma_i,w)| \leq K\|w\|_\varepsilon^2. \tag{3.18}$$

In the previous estimates, the constant K *does not depend on* $\varepsilon > 0$, $\theta \in \mathbb{R}^N$, $\sigma \in \Sigma_\delta$ *and* $w \in \mathcal{W}^\varepsilon_{\theta,\sigma}$.

The proof of this proposition is very technical. For the convenience of the reader, we postpone it in Section 4.
We aim to estimate the solution (θ,σ,w) of System (3.12)-(3.13)-(3.14). We will prove that the linear part of (3.14) is coercive, so that we will obtain that while $|\sigma(t) - \sigma^{ref}(t)| \leq \delta$ (that implies both that $\sigma(t)$ remains in Σ_δ and that h is constant in each wall), then w remains small. To conclude, we will observe that since w remains small, the system for θ_i and σ_i is a small perturbation of System (1.12) concerning $(\theta_i^{ref},\sigma_i^{ref})$, so that (θ,σ) remains close to $(\sigma^{ref},\theta^{ref})$.

Second step: study of Λ_ε.

As said before, the key point of our analysis is that the linear part of (3.14) is coercive. The coercivity of Λ_ε is deduced from the following proposition concerning the operator $\mathcal{H}^\varepsilon$.

Proposition 3.2. *There exists $\varepsilon_0 > 0$, there exist c_1 and c_2 such that for all $\theta \in I\!R^N$, $\sigma \in \Sigma_\delta$, $\varepsilon \le \varepsilon_0$ and $w \in \mathcal{W}^\varepsilon_{\theta,\sigma}$, the following estimates hold:*

$$\big\langle \mathcal{H}^\varepsilon(w)|w\big\rangle \le \frac{\varepsilon}{1-\tau}\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|^2_{L^2}$$

$$c_1\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2} \le \|\varepsilon\partial_{xx}w\|_{L^2} + \|\partial_x w\|_{L^2} + \frac{1}{\varepsilon}\|w\|_{L^2} \le c_2\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}$$

$$c_1\|w\|_\varepsilon \le \sqrt{\big\langle \mathcal{H}^\varepsilon(w)|w\big\rangle} \le c_2\|w\|_\varepsilon.$$

We recall the the constant $\tau > 0$ appears in Assumption (1.11) concerning the applied field h. The proof of this proposition is postponed in subsection 3.2.

Third step: variational estimates for w.

We take the inner product of (3.14) with $\mathcal{H}^\varepsilon(w)$. Using (3.17), we remark that we have

$$\frac{1}{2}\frac{d}{dt}\big\langle \mathcal{H}^\varepsilon(w)|w\big\rangle = \big\langle \frac{\partial w}{\partial t}|\mathcal{H}^\varepsilon(w)\big\rangle + \frac{1}{2}\big\langle \frac{df^\sigma_\varepsilon}{dt}w|w\big\rangle.$$

In addition,

$$\frac{df^\sigma_\varepsilon}{dt} = \sum_{i=1}^N \partial_{\sigma_i} f^\sigma_\varepsilon \frac{d\sigma_i}{dt},$$

so we obtain:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\big\langle \mathcal{H}^\varepsilon(w)|w\big\rangle + \|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|^2_{L^2} = \quad & \big\langle a_\varepsilon|\mathcal{H}^\varepsilon(w)\big\rangle + \big\langle P_\varepsilon w|\mathcal{H}^\varepsilon(w)\big\rangle + \big\langle l^\varepsilon w|\mathcal{H}^\varepsilon(w)\big\rangle \\ & + \frac{1}{2}\sum_{i=1}^N \frac{d\sigma_i}{dt}\big\langle \partial_{\sigma_i} f^\sigma_\varepsilon w|w\big\rangle + \big\langle G^\varepsilon(w,\theta,\sigma)|\mathcal{H}^\varepsilon(w)\big\rangle.\end{aligned}$$

Using (3.13) we obtain that

$$\frac{1}{2}\frac{d}{dt}\big\langle \mathcal{H}^\varepsilon(w)|w\big\rangle + \|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|^2_{L^2} \le M_1 + M_2 + M_3,$$

where

- $M_1 = \big\langle a_\varepsilon|\mathcal{H}^\varepsilon(w)\big\rangle$,
- $M_2 = \big\langle P_\varepsilon w|\mathcal{H}^\varepsilon(w)\big\rangle + \dfrac{1}{2}\displaystyle\sum_{i=1}^N \left((-1)^i h_i + a^{\sigma_i}_\varepsilon\right)\big\langle \partial_{\sigma_i} f^\sigma_\varepsilon w|w\big\rangle + \big\langle l^\varepsilon w|\mathcal{H}^\varepsilon(w)\big\rangle$,
- $M_3 = \displaystyle\sum_{i=1}^N \left(-h_i \Pi^{\sigma_i}_\varepsilon(w) + l^{\sigma_i}_\varepsilon(w) + G^{\sigma_i}_\varepsilon(\theta_i,\sigma_i,w)\right)\big\langle \partial_{\sigma_i} f^\sigma_\varepsilon w|w\big\rangle + \big\langle G^\varepsilon(w,\theta,\sigma)|\mathcal{H}^\varepsilon(w)\big\rangle$.

Since by Proposition 3.2, $\|\mathcal{H}^\varepsilon(w)\|_{L^2} \le C\,\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}$, we obtain by the Young Lemma that:

$$M_1 \le \frac{1}{\varepsilon}\,\|a_\varepsilon\|^2_{L^2} + C\varepsilon\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|^2_{L^2}.$$

The term M_2 coming from the linear terms induced by the applied magnetic field h, is estimated in the following proposition, proved in Section 3.3:

Proposition 3.3. *While $\sigma(t)$ remains in Σ_δ, then*

$$|M_2| \le \Big(\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}) + \|h\|_{L^\infty}(1 + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}))\Big)\,\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|^2_{L^2}.$$

Concerning the non linear part M_3, we remark that using Proposition 4.10 and Corollary 3.2, we obtain that

$$\left|\left\langle G^\varepsilon(w,\theta,\sigma)|\mathcal{H}^\varepsilon w\right\rangle\right| \leq K\|w\|_{L^\infty}\|\mathbf{m}_\varepsilon\times\mathcal{H}^\varepsilon(w)\|_{L^2}^2.$$

In addition, using the estimates in Proposition 3.1, since $\|\partial_{\sigma_i}f_\varepsilon^\sigma\|_{L^\infty}\leq\dfrac{c}{\varepsilon^2}$, we have

$$\begin{aligned}\left|(-h_i\Pi_\varepsilon^{\sigma_i}(w)+l_\varepsilon^{\sigma_i}(w)+G_\varepsilon^{\sigma_i}(\theta_i,\sigma_i,w))\left\langle\partial_{\sigma_i}f_\varepsilon^\sigma w|w\right\rangle\right| &\leq C\|w\|_\varepsilon\frac{1}{\varepsilon^2}\|w\|_{L^2}^2\\ &\leq C\|w\|_\varepsilon\|\mathbf{m}_\varepsilon\times\mathcal{H}^\varepsilon(w)\|_{L^2}^2\end{aligned}$$

using Proposition 3.2.

So we obtain that for ε small enough, while σ remains in Σ_δ, then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\langle\mathcal{H}^\varepsilon(w)|w\right\rangle+\|\mathbf{m}_\varepsilon\times\mathcal{H}^\varepsilon(w)\|_{L^2}^2 \leq\ & \frac{1}{\varepsilon}\|a_\varepsilon\|_{L^2}^2+\|\mathbf{m}_\varepsilon\times\mathcal{H}^\varepsilon(w)\|_{L^2}^2\Big(\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})+C\varepsilon+\\ & +\|h\|_{L^\infty}(1+\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}))+C\|w\|_\varepsilon\Big),\\ \leq\ & \frac{1}{\varepsilon}\|a_\varepsilon\|_{L^2}^2+\|\mathbf{m}_\varepsilon\times\mathcal{H}^\varepsilon(w)\|_{L^2}^2\left(1-2\tau+K\sqrt{\left\langle\mathcal{H}^\varepsilon(w)|w\right\rangle}\right),\end{aligned} \tag{3.19}$$

using Assumption (1.11) on the applied field, and by the last inequality in Proposition 3.2.

Fourth step: comparison lemma for w.

Equation (3.19) yields that for ε small enough, while σ remains in Σ_δ, then

$$\frac{1}{2}\frac{d}{dt}\left\langle\mathcal{H}^\varepsilon(w)|w\right\rangle+\|\mathbf{m}_\varepsilon\times\mathcal{H}^\varepsilon(w)\|_{L^2}^2\left(2\tau-K\sqrt{\left\langle\mathcal{H}^\varepsilon(w)|w\right\rangle}\right)\leq\frac{1}{\varepsilon}\|a_\varepsilon\|_{L^2}^2,$$

so while $\left\langle\mathcal{H}^\varepsilon(w)|w\right\rangle\leq\left(\dfrac{\tau}{K}\right)^2$, we have:

$$\frac{1}{2}\frac{d}{dt}\left\langle\mathcal{H}^\varepsilon(w)|w\right\rangle+\tau\|\mathbf{m}_\varepsilon\times\mathcal{H}^\varepsilon(w)\|_{L^2}^2\leq\frac{1}{\varepsilon}\|a_\varepsilon\|_{L^2}^2.$$

Using the first estimate claimed in Proposition 3.2, we obtain then that while $\left\langle\mathcal{H}^\varepsilon(w)|w\right\rangle\leq\left(\dfrac{\tau}{K}\right)^2$, we have:

$$\frac{1}{2}\frac{d}{dt}\left\langle\mathcal{H}^\varepsilon(w)|w\right\rangle+\frac{\tau(1-\tau)}{\varepsilon}\left\langle\mathcal{H}^\varepsilon(w)|w\right\rangle\leq\frac{1}{\varepsilon}\|a_\varepsilon\|_{L^2}^2.$$

We remark that $a_\varepsilon=\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$ so there exists a constant c such that

$$\forall\, t,\ \ \frac{1}{\varepsilon}\|a_\varepsilon\|_{L^2}^2\leq c^2e^{-\frac{\delta}{2\varepsilon}}\tau(1-\tau),$$

so that while σ remains in Σ_δ, while $\left\langle\mathcal{H}^\varepsilon(w)|w\right\rangle\leq\left(\dfrac{\tau}{K}\right)^2$,

$$\frac{1}{2}\frac{d}{dt}\left\langle\mathcal{H}^\varepsilon(w)|w\right\rangle+\frac{\tau(1-\tau)}{\varepsilon}\left\langle\mathcal{H}^\varepsilon(w)|w\right\rangle\leq c^2\tau(1-\tau)e^{-\frac{\delta}{2\varepsilon}}.$$

By comparison argument, we obtain finally that for ε small enough, while σ remains in Σ_δ, while $\left\langle\mathcal{H}^\varepsilon(w)|w\right\rangle\leq\left(\dfrac{\tau}{K}\right)^2$,

$$\begin{aligned}\left\langle\mathcal{H}^\varepsilon(w)|w\right\rangle(t)\leq\ & \left\langle\mathcal{H}^\varepsilon(w_0)|w_0\right\rangle e^{-2\frac{\tau(1-\tau)}{\varepsilon}t}+c^2\varepsilon e^{-\frac{\delta}{2\varepsilon}}\\ \leq\ & c_2^2\|w_0\|_\varepsilon^2e^{-2\frac{\tau(1-\tau)}{\varepsilon}t}+c^2\varepsilon e^{-\frac{\delta}{2\varepsilon}},\end{aligned}$$

that is, taking the square root:

$$\sqrt{\langle \mathcal{H}^\varepsilon(w)|w\rangle(t)} \le c_2\|w_0\|_\varepsilon e^{-\frac{\tau(1-\tau)}{\varepsilon}t} + c\sqrt{\varepsilon}e^{-\frac{\delta}{4\varepsilon}}. \tag{3.20}$$

We fix $\nu_0 > 0$. There exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$,

$$c\sqrt{\varepsilon}e^{-\frac{\delta}{4\varepsilon}} \le \inf\left\{\frac{\nu_0}{c_1}, \frac{1}{4}\frac{\tau}{K}\right\} \tag{3.21}$$

(we recall that the constants c_1 and c_2 appear in the norm equivalence inequality in Proposition 3.2).

We fix $\alpha_0 > 0$ such that

$$\frac{\alpha_0}{c_2} \le \inf\left\{\frac{\nu_0}{c_1}, \frac{1}{4}\frac{\tau}{K}\right\}.$$

Then for all $\varepsilon < \varepsilon_0$, if $\|w_0\|_\varepsilon \le \alpha_0$, then by Inequality (3.21), we obtain that while σ remains in Σ_δ, $\langle \mathcal{H}^\varepsilon(w)|w\rangle(t)$ cannot become greater than the bound $\left(\frac{\tau}{K}\right)^2$ so that (3.20) remains valid, and in particular, using Proposition 3.2,

$$\|w(t)\|_\varepsilon \le \nu_0.$$

To summarize, we have prove that if $\nu_0 > 0$ is fixed, there exists ε_0 and α_0 such that for all ε such that $0 < \varepsilon < \varepsilon_0$, if $\|w_0\|_\varepsilon \le \alpha_0$, then while σ remains in Σ_δ, we have

$$\|w(t)\|_\varepsilon \le \nu_0 \text{ and } \|w(t)\|_\varepsilon \le \frac{c_2}{c_1}\|w_0\|_\varepsilon e^{-\frac{\tau(1-\tau)}{\varepsilon}t} + \frac{c}{c_1}\sqrt{\varepsilon}e^{-\frac{\delta}{4\varepsilon}}.$$

Last step: estimates on θ and σ.

From (1.12) and (3.12)-(3.13), while σ remains in Σ_δ, we have

$$\frac{d}{dt}(\theta_i - \theta_i^{ref}) = a_\varepsilon^{\theta_i} - h_i\Pi_\varepsilon^{\theta_i}(w) + l_\varepsilon^{\theta_i}(w) + G_\varepsilon^{\theta_i}(\theta_i, \sigma_i, w),$$

$$\tfrac{d}{dt}(\sigma_i - \sigma_i^{ref}) = a_\varepsilon^{\sigma_i} - h_i\Pi_\varepsilon^{\sigma_i}(w) + l_\varepsilon^{\sigma_i}(w) + G_\varepsilon^{\sigma_i}(\theta_i, \sigma_i, w).$$

So Using (3.16) and (3.18), using that $a_\varepsilon^{\theta_i} = \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$ and that $a_\varepsilon^{\sigma_i} = \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$, we obtain that there exists a constant C such that while σ remains in Σ_δ,

$$\left|\frac{d}{dt}(\theta_i - \theta_i^{ref})\right| + \left|\frac{d}{dt}(\sigma_i - \sigma_i^{ref})\right| \le C\sqrt{\varepsilon}e^{-\frac{\delta}{4\varepsilon}} + C\|w(t)\|_\varepsilon. \tag{3.22}$$

We assume now as in the previous step that $0 < \varepsilon < \varepsilon_0$ and $\|w_0\|_\varepsilon \le \alpha_0$. We obtain then that while σ remains in Σ_δ,

$$\left|\frac{d}{dt}(\theta_i - \theta_i^{ref})\right| + \left|\frac{d}{dt}(\sigma_i - \sigma_i^{ref})\right| \le C\sqrt{\varepsilon}e^{-\frac{\delta}{4\varepsilon}} + C\frac{c_2}{c_1}\|w_0\|_\varepsilon e^{-\frac{\tau(1-\tau)}{\varepsilon}t} + C\frac{c}{c_1}\sqrt{\varepsilon}e^{-\frac{\delta}{4\varepsilon}},$$

which yields by integration that while σ remains in Σ_δ,

$$\left|\theta_i(t) - \theta_i^{ref}(t)\right| \le \left|\theta_i(0) - \theta_i^{ref}(0)\right| + C\frac{c_2}{c_1}\frac{\varepsilon}{\tau(1-\tau)}\|w_0\|_\varepsilon + C(1+\frac{c}{c_1})\sqrt{\varepsilon}e^{-\frac{\delta}{4\varepsilon}}t, \tag{3.23}$$

$$\left|\sigma_i(t) - \sigma_i^{ref}(t)\right| \le \left|\sigma_i(0) - \sigma_i^{ref}(0)\right| + C\frac{c_2}{c_1}\frac{\varepsilon}{\tau(1-\tau)}\|w_0\|_\varepsilon + C(1+\frac{c}{c_1})\sqrt{\varepsilon}e^{-\frac{\delta}{4\varepsilon}}t. \tag{3.24}$$

Let $\varepsilon_1 > 0$ such that $\varepsilon_1 \le \varepsilon_0$ and satisfying $C(1+\frac{c}{c_1})\sqrt{\varepsilon_1} \le \inf\left\{\frac{\delta}{4}, \frac{\nu_0}{3}\right\}$.

Let $\alpha_1 > 0$ such that $\alpha_1 \le \inf\left\{\alpha_0, \frac{\delta}{4}, \frac{\nu_0}{3}\right\}$ and such that $C\frac{c_2}{c_1}\frac{\varepsilon_1}{\tau(1-\tau)}\alpha_1 \le \inf\left\{\frac{\delta}{4}, \frac{\nu_0}{3}\right\}$.

Let us assume that $\varepsilon < \varepsilon_1$ and that the initial data satisfy $\|w_0\|_\varepsilon \leq \alpha_1$, $|\theta_i(0) - \theta_i^{ref}(0)| \leq \alpha_1$ and $|\sigma_i(0) - \sigma_i^{ref}(0)| \leq \alpha_1$.
Then from the last step, we know that while σ remains in Σ_δ, then $\|w(t)\|_\varepsilon \leq \nu_0$.
In addition, from (3.23) and (3.24), we obtain that while σ remains in Σ_δ, while $t \leq e^{\frac{\delta}{4\varepsilon}}$, then $\left|\theta_i(t) - \theta_i^{ref}(t)\right| \leq \nu_0$, $\left|\sigma_i(t) - \sigma_i^{ref}(t)\right| \leq \nu_0$ and $\left|\sigma_i(t) - \sigma_i^{ref}(t)\right| \leq \frac{3\delta}{4}$, so that σ remains in Σ_δ, which implies that the previous estimates are valid in the whole time interval $[0, e^{\frac{\delta}{4\varepsilon}}]$.

This concludes the proof of Theorem 1.3.

□

3.2 Coercivity of the operator $\mathcal{H}^\varepsilon$

We aim to prove the following result:

Proposition 3.4. *There exists $\varepsilon_0 > 0$ such that for all $\theta \in I\!R^N$, for all $\sigma \in \Sigma_\delta$, for all $\varepsilon > 0$, for all $w \in \mathcal{W}^\varepsilon_{\theta,\sigma}$,*

$$\langle \mathcal{H}^\varepsilon(w)|w \rangle \geq \frac{1-\tau}{\varepsilon}\|w\|^2_{L^2}.$$

Remark 3.1. *The real $\tau > 0$ is given* a priori *by the bound on the applied field h in Assumption (1.13).*

On the one hand, for a configuration without any wall, *i.e.* with only one domain, this inequality is straightforward. Indeed, in this case, $\mathcal{W}^\varepsilon_{\theta,\sigma}$ reduces to the set:

$$\left\{w \in H^2([-L, L]; I\!R^3), w_1 = 0\right\},$$

and

$$\mathcal{H}^\varepsilon(w) = -\varepsilon\partial_{xx} w + \frac{1}{\varepsilon} w,$$

thus

$$\langle \mathcal{H}^\varepsilon(w)|w \rangle \geq \frac{1}{\varepsilon}\|w\|^2_{L^2}.$$

On the other hand, for a configuration of one wall in an infinite wire, the coercivity is obtained in [14] by describing this operator in a moving frame which takes into account that $w(t, x)$ remains orthogonal to $\mathbf{m}_\varepsilon(\theta(t), \sigma(t))(x)$.

We gather both coercivity properties, the first one in the domains, the second one in the walls, by using a localization argument called IMS formula (see [18] and the references therein). The key point of this formula is the use of a convenient system of cut-off functions.

Let us detail the previous points.

Proof of Proposition 3.4.

First Step: convenient system of cut off functions.

We assume that σ is fixed in Σ_δ. We introduce a system of cut off functions $\chi_0, \chi_1, \dots, \chi_N$ such that

- $\chi_i \in \mathcal{C}^\infty$,
- supp $\chi_0 \in [0, L] \setminus \bigcup_{i=1}^{N} [\sigma_i - \delta/2, \sigma_i + \delta/2]$ (χ_0 is localized in the domains),
- supp $\chi_i \subset [\sigma_i - \frac{3\delta}{4}, \sigma_i + \frac{3\delta}{4}]$ for $i \neq 0$ (χ_i is localized in the i^{th}wall), (3.25)
- $\sum_{i=0}^{N} (\chi_i)^2 = 1.$

We can assume that there exists a constant K_δ, only depending on δ but not on $\sigma \in \Sigma_\delta$ such that

$$\|\chi_0'\|_{L^\infty} + \ldots + \|\chi_N'\|_{L^\infty} + \|\chi_0''\|_{L^\infty} + \ldots + \|\chi_N''\|_{L^\infty} \leq K_\delta. \tag{3.26}$$

We remark that χ_0 and χ_i can be simultaneously non zero only in the zone $[\sigma_i - 3\delta/4, \sigma_i - \delta/2] \cup [\sigma_i + \delta/2, \sigma_i + 3\delta/4]$, *i.e.* in the transitional zones.

We have:

$$\begin{aligned}\langle \partial_x(\chi_j w) | \partial_x(\chi_j w) \rangle &= -\langle \partial_x \left(\chi_j \partial_x w + (\partial_x \chi_j) w \right) | \chi_j w \rangle \\ &= -\langle 2\partial_x \chi_j \partial_x w | \chi_j w \rangle - \langle \chi_j \partial_{xx} w | \chi_j w \rangle - \langle (\partial_{xx}\chi_j) w | \chi_j w \rangle.\end{aligned}$$

Therefore

$$-\langle \partial_{xx} w | (\chi_j)^2 w \rangle = \langle \partial_x(\chi_j w) | \partial_x(\chi_j w) \rangle + \langle \partial_x(\chi_j^2) \partial_x w | w \rangle + \langle \partial_{xx}\chi_j w | \chi_j w \rangle.$$

We sum up these equalities for $j = 0$ to $j = N$. We remark that

$$\sum_{j=0}^{N} \partial_x(\chi_j^2) = \partial_x \sum_{j=0}^{N} (\chi_j^2) = 0,$$

so we obtain that:

$$-\sum_{j=0}^{N} \langle \partial_{xx} w | (\chi_j)^2 w \rangle = \sum_{j=0}^{N} \langle \partial_x(\chi_j w) | \partial_x(\chi_j w) \rangle + \sum_{j=0}^{N} \langle \partial_{xx}\chi_j w | \chi_j w \rangle.$$

So,

$$\langle \mathcal{H}^\varepsilon(w) | w \rangle = \sum_{j=0}^{N} \langle \mathcal{H}^\varepsilon(\chi_j w) | \chi_j w \rangle + \varepsilon \sum_{j=0}^{N} \langle \partial_{xx}\chi_j w | \chi_j w \rangle. \tag{3.27}$$

Using the bounds (3.26) on the cut off functions, we estimate the right hand side term as follows:

$$\left| \varepsilon \sum_{j=0}^{N} \langle \partial_{xx}\chi_j w | \chi_j w \rangle \right| \leq K\varepsilon \|w\|_{L^2}^2, \tag{3.28}$$

where the constant K only depends on δ.

Second Step: coercivity in the domains.

The estimate in the domains is given by the following lemma:

Lemma 3.1. *There exists ε_0 only depending on δ such that for all $\varepsilon < \varepsilon_0$, for all $\sigma \in \Sigma_\delta$, if χ_0 satisfies (3.25), then*

$$\forall\, \theta \in I\!R^N, \forall\, w \in \mathcal{W}_{\theta,\sigma}^\varepsilon,\ \langle \mathcal{H}^\varepsilon(\chi_0 w) | \chi_0 w \rangle \geq \frac{1}{\varepsilon} \|\chi_0 w\|_{L^2}^2 - \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}) \|w\|_{L^2}^2$$

Proof. On the support of χ_0, we have

$$f_\varepsilon^\sigma = \frac{1}{\varepsilon} + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}).$$

In addition, on $[\sigma_i + \frac{\delta}{2}, \sigma_{i+1} - \frac{\delta}{2}]$, $\mathbf{m}_\varepsilon(x) = (-1)^i e_1 + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$, so $0 = \mathbf{m}_\varepsilon(x) \cdot w(x) = (-1)^i w_1 + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}) \cdot w$, that is

$$\|\chi_0 w_1\|_{L^2} \leq \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}) \|\chi_0 w\|_{L^2}.$$

So,

$$\langle \mathcal{H}^\varepsilon(\chi_0 w) | \chi_0 w \rangle = \varepsilon \|\partial_x(\chi_0 w)\|_{L^2}^2 + \frac{1}{\varepsilon} \|\chi_0 w\|_{L^2}^2 + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}) \|\chi_0\|_{L^2}^2,$$

hence we obtain the claimed estimate. □

Third Step: coercivity for one wall.

The estimate in the walls is given by the following lemma:

Lemma 3.2. *There exists ε_0 only depending on δ such that for all $\varepsilon < \varepsilon_0$, for all $\sigma \in \Sigma_\delta$, for all $j \in \{1, \ldots, N\}$, for χ_j satisfying (3.25), then*

$$\forall\, \theta \in I\!R^N, \forall\, w \in \mathcal{W}^\varepsilon_{\theta,\sigma},\ \left\langle \mathcal{H}^\varepsilon(\chi_j w) | \chi_j w \right\rangle \geq \frac{1}{\varepsilon}\|\chi_j w\|^2_{L^2} - \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})\|w\|^2_{L^2}.$$

Proof. In the wall i, that is in the zone $[\sigma_i - \frac{3\delta}{4}, \sigma_i + \frac{3\delta}{4}]$, we recall that

$$\mathbf{m}_\varepsilon(\theta,\sigma)(x) = (-1)^{i+1}\mathbf{R}_{\frac{\theta_i}{\varepsilon}} M_0(\frac{x-\sigma_i}{\varepsilon}),$$

so that:

$$\partial_{\sigma_i}\mathbf{m}_\varepsilon(\theta,\sigma)(x) = \frac{(-1)^{i+1}}{\varepsilon\cosh\frac{x-\sigma_i}{\varepsilon}}\mathbf{R}_{\frac{\theta_i}{\varepsilon}} M_1(\frac{x-\sigma_i}{\varepsilon}),$$

$$\partial_{\theta_i}\mathbf{m}_\varepsilon(\theta,\sigma)(x) = \frac{(-1)^{i+1}}{\varepsilon\cosh\frac{x-\sigma_i}{\varepsilon}}\mathbf{R}_{\frac{\theta_i}{\varepsilon}} M_2,$$

where

$$M_0(z) = \begin{pmatrix} \tanh z \\ 1/\cosh z \\ 0 \end{pmatrix}, \quad M_1(z) = \begin{pmatrix} -1/\cosh z \\ \tanh z \\ 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Using that $\chi_i w \cdot \mathbf{m}_\varepsilon = 0$, we describe $\chi_i w$ as:

$$(\chi_i w)(x) = r_1(\frac{x-\sigma_i}{\varepsilon})\mathbf{R}_{\frac{\theta_i}{\varepsilon}} M_1(\frac{x-\sigma_i}{\varepsilon}) + r_2(\frac{x-\sigma_i}{\varepsilon})\mathbf{R}_{\frac{\theta_i}{\varepsilon}} M_2 \tag{3.29}$$

where $r = (r_1, r_2) \in H^2(I\!R)$ is compactly supported in $[-\frac{3\delta}{4\varepsilon}, \frac{3\delta}{4\varepsilon}]$.

From the condition $\left\langle w | \partial_{\sigma_i}\mathbf{m}_\varepsilon \right\rangle = 0$, we deduce that $\left\langle \chi_i w | \partial_{\sigma_i}\mathbf{m}_\varepsilon \right\rangle = \left\langle (\chi_i - 1) w | \partial_{\sigma_i}\mathbf{m}_\varepsilon \right\rangle$, so

$$|\left\langle \chi_i w | \partial_{\sigma_i}\mathbf{m}_\varepsilon \right\rangle| \leq \|w\|_{L^2}\|(\chi_i - 1)\partial_{\sigma_i}\mathbf{m}_\varepsilon\|_{L^2} \leq \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})\|w\|_{L^2},$$

since $\chi_i - 1 = 0$ in $[\sigma_i - \frac{\delta}{2}, \sigma_i + \frac{\delta}{2}]$ and since $\partial_{\sigma_i}\mathbf{m}_\varepsilon$ is exponentially small outside this interval.

Now, using (3.29),

$$\left\langle \chi_i w | \partial_{\sigma_i}\mathbf{m}_\varepsilon \right\rangle = (-1)^{i+1}\int_{I\!R} r_1(\frac{x-\sigma_i}{\varepsilon})\frac{1}{\varepsilon\cosh\frac{x-\sigma_i}{\varepsilon}}dx = \int_{I\!R} r_1(z)\frac{1}{\cosh z}dz = \left\langle r_1 | \frac{1}{\cosh z}\right\rangle_{I\!R},$$

where we denote by $\left\langle .|. \right\rangle_{I\!R}$ the L^2 inner product on $I\!R$.

So,

$$\left\langle r_1 | \frac{1}{\cosh z}\right\rangle_{I\!R} = \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})\|w\|_{L^2}. \tag{3.30}$$

In the same way we prove that

$$\left\langle r_2 | \frac{1}{\cosh z}\right\rangle_{I\!R} = \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})\|w\|_{L^2}. \tag{3.31}$$

By a tedious calculation using Subsection 2.1, we obtain that

$$\begin{aligned}\mathcal{H}^\varepsilon(\chi w) = \quad & \left(-\frac{2}{\varepsilon}\partial_z r_1(\frac{x-\sigma_i}{\varepsilon})\frac{1}{\cosh\frac{x-\sigma_i}{\varepsilon}} - \frac{(-1)^i}{\varepsilon}\frac{\sinh\frac{x-\sigma_i}{\varepsilon}}{\cosh^2\frac{x-\sigma_i}{\varepsilon}} r_1(\frac{x-\sigma_i}{\varepsilon})\right)\mathbf{R}_{\frac{\theta_i}{\varepsilon}} M_0(\frac{x-\sigma_i}{\varepsilon}) \\ & + \left(-\frac{1}{\varepsilon}\partial_{zz} r_1(\frac{x-\sigma_i}{\varepsilon}) + \frac{1}{\varepsilon}f(\frac{x-\sigma_i}{\varepsilon}) r_1(\frac{x-\sigma_i}{\varepsilon})\right)\mathbf{R}_{\frac{\theta_i}{\varepsilon}} M_1(\frac{x-\sigma_i}{\varepsilon}) \\ & + \left(-\frac{1}{\varepsilon}\partial_{zz} r_2(\frac{x-\sigma_i}{\varepsilon}) + \frac{1}{\varepsilon}f(\frac{x-\sigma_i}{\varepsilon}) r_2(\frac{x-\sigma_i}{\varepsilon})\right)\mathbf{R}_{\frac{\theta_i}{\varepsilon}} M_2,\end{aligned} \tag{3.32}$$

where $f(z)=2\tanh^2 z-1$.
Therefore,

$$\big\langle \mathcal{H}^\varepsilon(\chi_i w)|\chi_i w\big\rangle = \big\langle \mathcal{L}r_1|r_1\big\rangle_{I\!R} + \big\langle \mathcal{L}r_2|r_2\big\rangle_{I\!R}$$

where

$$\mathcal{L}r_j = -\partial_{zz}r_j + f(z)r_j.$$

We recall the properties of the operator $\mathcal{L}$ (see [9] and [14]):

Proposition 3.5. *The operator $\mathcal{L}$ with domain $H^2(I\!R)$ is self-adjoint and positive, 0 in a simple eigenvalue of $\mathcal{L}$ associated to the eigenvector $\dfrac{1}{\cosh x}$. In the orthogonal of its Kernel, $\mathcal{L}$ satisfies:*

$$\text{if } \langle u|\frac{1}{\cosh z}\rangle_{I\!R}=0, \quad \big\langle \mathcal{L}u|u\big\rangle_{I\!R} \geq \|u\|^2_{L^2(I\!R)}, \tag{3.33}$$

and

$$\text{if } \langle u|\frac{1}{\cosh z}\rangle_{I\!R}=0, \quad \|r\|_{L^2(I\!R)} \leq \|\ell(r)\|_{L^2(I\!R)} \leq \|\mathcal{L}(r)\|_{L^2(I\!R)}, \tag{3.34}$$

where ℓ is given by $\ell=\partial_x+\tanh x$ and where we denote by $\|\|_{L^2(I\!R)}$ the $L^2(I\!R)$ usual norm.

Proof. Since $\mathcal{L}=\ell^*\circ\ell$ with $\ell=\partial_x+\tanh x$, $\mathcal{L}$ is a positive self-adjoint operator.

In addition, by a simple calculation, 0 in a simple eigenvalue of $\mathcal{L}$ associated to the eigenvector $\dfrac{1}{\cosh x}$. By standard results, the essential spectrum of $\mathcal{L}$ is $[1,+\infty[$.
We remark now that $\ell\circ\ell^*=-\partial_{xx}+1$, so $\mathcal{L}$ does not have anymore eigenvalue since $\mathcal{L}r=\lambda r$ induces that $\ell\circ\mathcal{L}r=\lambda\ell r$, so since $\mathcal{L}=\ell^*\circ\ell$, we obtain that $(-\partial_{xx}+1)\ell r=\lambda\ell r$ wich implies that $\ell r=0$ since $-\partial_{xx}+1$ does not have eigenvalues.
So, on $\big(\frac{1}{\cosh x}\big)^\perp$, $\mathcal{L}\geq 1$ and the inequalities in (3.34) and in (3.33) follow.

□

We project r_1 on $(\dfrac{1}{\cosh z})^\perp$ writing $r_1=\bar{r}_1+r_1^\perp$, with $r_1^\perp=\dfrac{1}{2}\langle r_1|\dfrac{1}{\cosh z}\rangle_{I\!R}\dfrac{1}{\cosh z}$ (we recall that $\displaystyle\int_{I\!R}\frac{1}{\cosh^2 z}=2$). Since $\mathcal{L}r_1=\mathcal{L}\bar{r}_1$, we have:

$$\begin{aligned}
\big\langle \mathcal{L}r_1|r_1\big\rangle_{I\!R} = &\ \big\langle \mathcal{L}\bar{r}_1|\bar{r}_1\big\rangle_{I\!R}\\
\geq &\ \|\bar{r}_1\|^2_{L^2(I\!R)}\\
\geq &\ \|r_1\|^2_{L^2(I\!R)} - \|r_1^\perp\|^2_{L^2(I\!R)}\\
\geq &\ \|r_1\|_{L^2(I\!R)}\|^2 - \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})\|w\|^2_{L^2} \text{ from (3.31).}
\end{aligned}$$

In the same way we prove that

$$\big\langle \mathcal{L}r_2|r_2\big\rangle_{I\!R} \geq \|r_2\|^2_{L^2(I\!R)} - \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})\|w\|^2_{L^2}.$$

Therefore,

$$\big\langle \mathcal{H}^\varepsilon(\chi_i w)|\chi_i w\big\rangle \geq \|r_1\|^2_{L^2(I\!R)} + \|r_2\|^2_{L^2(I\!R)} - \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})\|w\|^2_{L^2}$$

and since by rescaling,

$$\|r_1\|^2_{L^2(I\!R)} + \|r_2\|^2_{L^2(I\!R)} = \frac{1}{\varepsilon}\|\chi_i w\|^2_{L^2},$$

we conclude the proof of Lemma 3.2. □

Last step: end of the proof of Proposition 3.4.
We first remark that, from (3.25),

$$\|w\|_{L^2}^2 = \sum_{i=0}^{N} \|\chi_i w\|_{L^2}^2. \tag{3.35}$$

Lemmas 3.1 and 3.2 together with (3.27), (3.28) and (3.35) yield

$$\langle \mathcal{H}^\varepsilon(w)|w\rangle \geq \frac{1}{\varepsilon}\|w\|_{L^2}^2 - K\varepsilon\|w\|_{L^2}^2$$

so, if ε is small enough,

$$\langle \mathcal{H}^\varepsilon(w)|w\rangle \geq \frac{1-\tau}{\varepsilon}\|w\|_{L^2}^2$$

This concludes the proof of Proposition 3.4. □

From this result, we first obtain that the norms $\|\cdot\|_\varepsilon$ and $\langle \mathcal{H}^\varepsilon(w)|w\rangle^{\frac{1}{2}}$ are equivalent on $\mathcal{W}^\varepsilon_{\theta,\sigma}$:

Corollary 3.1. *There exist c_1 and c_2 such that for all $\varepsilon > 0$, for all $\sigma \in \Sigma_\delta$, $\theta \in I\!R^N$, $w \in \mathcal{W}^\varepsilon_{\theta,\sigma}$, then*

$$c_1\|w\|_\varepsilon \leq \sqrt{\langle \mathcal{H}^\varepsilon(w)|w\rangle} \leq c_2\|w\|_\varepsilon.$$

In addition, here exists K such that

$$\|w\|_{L^\infty} \leq K\left(\langle w|\mathcal{H}^\varepsilon(w)\rangle\right)^{\frac{1}{2}}.$$

In addition we have the following estimates:

Corollary 3.2. *There exists $\varepsilon_0 > 0$, there exists K such that for all $\theta \in I\!R^N$, $\sigma \in \Sigma_\delta$, $\varepsilon \leq \varepsilon_0$ and $w \in \mathcal{W}^\varepsilon_{\theta,\sigma}$, we have the following estimates:*

$$\|w\|_{L^2} \leq \frac{\varepsilon}{1-\tau}\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2},$$

$$\langle \mathcal{H}^\varepsilon(w)|w\rangle \leq \frac{\varepsilon}{1-\tau}\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}^2,$$

$$\|\varepsilon\partial_{xx}w\|_{L^2} + \|\partial_x w\|_{L^2} \leq K\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}.$$

Proof. We remark first that since $\mathbf{m}_\varepsilon \cdot w = 0$ and since $|\mathbf{m}_\varepsilon| = 1$, then $w = -\mathbf{m}_\varepsilon \times (\mathbf{m}_\varepsilon \times w)$ so

$$\langle w|\mathcal{H}^\varepsilon(w)\rangle = -\langle \mathbf{m}_\varepsilon \times (\mathbf{m}_\varepsilon \times w)|\mathcal{H}^\varepsilon(w)\rangle = \langle \mathbf{m}_\varepsilon \times w|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\rangle.$$

So, from Proposition 3.4, with Cauchy-Schwartz inequality, we obtain that

$$\begin{aligned}\|w\|_{L^2}^2 &\leq \frac{\varepsilon}{1-\tau}\langle w|\mathcal{H}^\varepsilon(w)\rangle \\ &\leq \frac{\varepsilon}{1-\tau}\langle \mathbf{m}_\varepsilon \times w|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\rangle \\ &\leq \frac{\varepsilon}{1-\tau}\|\mathbf{m}_\varepsilon \times w\|_{L^2}\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}.\end{aligned}$$

Now since $\mathbf{m}_\varepsilon \cdot w = 0$ and since $|\mathbf{m}_\varepsilon| = 1$, then $|w| = |\mathbf{m}_\varepsilon \times w|$, so $\|\mathbf{m}_\varepsilon \times w\|_{L^2} = \|w\|_{L^2}$ so

$$\|w\|_{L^2} \leq \frac{\varepsilon}{1-\tau}\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}.$$

From this estimate, we have:

$$\begin{aligned}\langle w|\mathcal{H}^\varepsilon(w)\rangle = & \ \langle \mathbf{m}_\varepsilon \times w|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\rangle \\ \leq & \ \|w\|_{L^2}\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2} \\ \leq & \ \frac{\varepsilon}{1-\tau}\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}^2 .\end{aligned}$$

Furthermore, since $\mathbf{m}_\varepsilon \times (\mathbf{m}_\varepsilon \times \partial_{xx} w) = \mathbf{m}_\varepsilon \cdot \partial_{xx} w\, \mathbf{m}_\varepsilon - \partial_{xx} w$, derivating twice the equality $\mathbf{m}_\varepsilon \cdot w = 0$ we obtain that

$$\begin{aligned}\varepsilon\partial_{xx} w = & \ -\varepsilon(2\partial_x w \cdot \partial_x \mathbf{m}_\varepsilon + w \cdot \partial_{xx}\mathbf{m}_\varepsilon)\mathbf{m}_\varepsilon - \mathbf{m}_\varepsilon \times (\mathbf{m}_\varepsilon \times \varepsilon\partial_{xx} w) \\ = & \ \mathbf{m}_\varepsilon \times (\mathbf{m}_\varepsilon \times (\mathcal{H}^\varepsilon(m) + \frac{1}{\varepsilon} w_1 e_1 - f_\varepsilon^\sigma \mathbf{m}_\varepsilon)) - \varepsilon(2\partial_x w \cdot \partial_x \mathbf{m}_\varepsilon + w \cdot \partial_{xx}\mathbf{m}_\varepsilon)\mathbf{m}_\varepsilon .\end{aligned}$$

So, since $\|\partial_x \mathbf{m}_\varepsilon\|_{L^\infty} \leq \dfrac{K}{\varepsilon}$ and since $\|\partial_{xx}\mathbf{m}_\varepsilon\|_{L^\infty} \leq \dfrac{K}{\varepsilon^2}$, we obtain that

$$\|\varepsilon\partial_{xx} w\|_{L^2} \leq K\|w \times \mathcal{H}^\varepsilon(w)\|_{L^2} + \frac{K}{\varepsilon}\|w\|_{L^2} + K\|\partial_x w\|_{L^2}.$$

By integration by parts,

$$\|\partial_x w\|_{L^2}^2 \leq \|\frac{1}{\varepsilon} w\|_{L^2}\|\varepsilon\partial_{xx} w\|_{L^2}, \tag{3.36}$$

so using the Young inequality,

$$\|\varepsilon\partial_{xx} w\|_{L^2} \leq K\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2} + \frac{K}{\varepsilon}\|w\|_{L^2} + \frac{1}{2}\|\varepsilon\partial_{xx} w\|_{L^2}.$$

By absorbing $\|\varepsilon\partial_{xx} w\|_{L^2}$, by using the previous inequality, we obtain that there exists K such that

$$\|\varepsilon\partial_{xx} w\|_{L^2} \leq K\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}.$$

This inequality together with (3.36) and with Sobolev inequalities yield estimates on $\|\partial_x w\|_{L^2}$ and $\|w\|_{L^\infty}$. □

3.3 Estimate of the remainder linear contributions

In order to prove Proposition 3.3, we aim to estimate M_2 given by:

$$M_2 = \langle P_\varepsilon w|\mathcal{H}^\varepsilon(w)\rangle + \frac{1}{2}\sum_{i=1}^N \left((-1)^i h_i + a_\varepsilon^{\sigma_i}\right)\langle \partial_{\sigma_i} f_\varepsilon^\sigma w|w\rangle + \langle l^\varepsilon w|\mathcal{H}^\varepsilon(w)\rangle,$$

where

$$P_\varepsilon w = -\frac{h}{\varepsilon} w \times e_1 - \frac{h}{\varepsilon} w_1 \mathbf{m}_\varepsilon - \frac{h}{\varepsilon}\sin\varphi_\varepsilon^\sigma w + h\varepsilon\Pi_\varepsilon^{\theta_i}(w)\partial_{\theta_i}\mathbf{m}_\varepsilon + h\varepsilon\Pi_\varepsilon^{\sigma_i}(w)\partial_{\sigma_i}\mathbf{m}_\varepsilon .$$

The term $a_\varepsilon^{\sigma_i}$ is of order $\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$, and since $\|\partial_{\sigma_i} f_\varepsilon^\sigma\|_{L^\infty} \leq \dfrac{c}{\varepsilon^2}$, using Proposition 3.2 we obtain:

$$\left|\sum_{i=1}^N a_\varepsilon^{\sigma_i}\langle \partial_{\sigma_i} f_\varepsilon^\sigma w|w\rangle\right| \leq \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}^2.$$

As it is already remarked in Section 3.1,

$$\langle \partial_{\sigma_i}\mathbf{m}_\varepsilon|\mathcal{H}^\varepsilon(w)\rangle = \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}),$$

and

$$\langle \partial_{\theta_i}\mathbf{m}_\varepsilon|\mathcal{H}^\varepsilon(w)\rangle = \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}).$$

Hence,

$$\langle P_\varepsilon w|\mathcal{H}_e w\rangle = P_1 + P_2 + P_3 + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}),$$

with

- $P_1 = \Big\langle -\dfrac{h}{\varepsilon} w \times e_1 | \mathcal{H}^\varepsilon(w) \Big\rangle$,
- $P_2 = \Big\langle -\dfrac{h}{\varepsilon} w_1 \mathbf{m}_\varepsilon | \mathcal{H}^\varepsilon(w) \Big\rangle$,
- $P_3 = \Big\langle \dfrac{h}{\varepsilon} \sin \varphi_\varepsilon^\sigma w | \mathcal{H}^\varepsilon(w) \Big\rangle$.

First Step: estimate of P_1 and P_2.

Concerning the first term:

$$P_1 = \langle \frac{h}{\varepsilon} w \times e_1 | \varepsilon \partial_{xx} w \rangle$$

since $w \times e_1$ is orthogonal to e_1 and to $f_\varepsilon^\sigma w$. By integration by parts, we obtain that

$$P_1 = -\big\langle (\partial_x h) w \times e_1 | \partial_x w \big\rangle,$$

(the term $\big\langle h\, \partial_x w \times e_1 | \partial_x w \big\rangle$ vanishes) so

$$|P_1| \leq \|\partial_x h\|_{L^\infty} \|w\|_{L^2} \|\partial_x w\|_{L^2} \leq K\sqrt{\varepsilon} \|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}^2. \tag{3.37}$$

The second term is estimated as follows:

$$\begin{aligned} P_2 &= -\frac{1}{\varepsilon}\big\langle \mathcal{H}^\varepsilon(w_1 \mathbf{m}_\varepsilon) | w \big\rangle \\ &= -\frac{1}{\varepsilon}\big\langle w_1 \mathcal{H}^\varepsilon(\mathbf{m}_\varepsilon) | w \big\rangle - 2\big\langle \partial_x w_1 \partial_x \mathbf{m}_\varepsilon | w \big\rangle - \big\langle \partial_{xx} w_1 \mathbf{m}_\varepsilon | w \big\rangle \\ &= -\tfrac{1}{\varepsilon}\big\langle w_1 \mathcal{H}^\varepsilon(\mathbf{m}_\varepsilon) | w \big\rangle - 2\big\langle \partial_x w_1 \partial_x \mathbf{m}_\varepsilon | w \big\rangle \\ &\quad \text{since } \mathbf{m}_\varepsilon \cdot w = 0. \end{aligned}$$

In the domains, $\partial_x \mathbf{m}_\varepsilon = 0$ and in the wall i, $\partial_x \mathbf{m}_\varepsilon = -\partial_{\sigma_i} \mathbf{m}_\varepsilon$, so that

$$P_2 = 2 \sum_{i=1}^N h_i \big\langle \partial_{\sigma_i} \mathbf{m}_\varepsilon \partial_x w_1 | w \big\rangle - \frac{1}{\varepsilon} \big\langle w_1 \mathcal{H}^\varepsilon(\mathbf{m}_\varepsilon) | w \big\rangle.$$

On the one hand, in the wall i,

$$\partial_{\sigma_i} \mathbf{m}_\varepsilon = (-1)^i \frac{1}{\varepsilon} \mathbf{m}_\varepsilon \times (\mathbf{m}_\varepsilon \times e_1) + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}),$$

hence

$$\partial_{\sigma_i} \mathbf{m}_\varepsilon \cdot w = (-1)^{i+1} \frac{1}{\varepsilon} e_1 \cdot w.$$

So,

$$\big\langle \partial_{\sigma_i} \mathbf{m}_\varepsilon \partial_x w_1 | w \big\rangle = (-1)^{i+1} \frac{1}{\varepsilon} \int_{\sigma_i - \delta}^{\sigma_i + \delta} \partial_x w_1 w_1 + \frac{1}{\varepsilon} \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}) = \frac{1}{\varepsilon} \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$$

as

$$\int_{\sigma_i - \delta}^{\sigma_i + \delta} \partial_x w_1 w_1 = \int_{\sigma_i - \delta}^{\sigma_i + \delta} \partial_x (w_1)^2 = 0,$$

since $w_1 = 0$ at the points $\sigma_i - \delta$ and $\sigma_i + \delta$.
On the other hand, we know from Proposition 2.1 that

$$|| \leq \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}) \|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}^2.$$

Therefore, we obtain that

$$|P_2| \leq \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}) \|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}^2. \tag{3.38}$$

Second Step: IMS formula.

It remains to estimate

$$I := P_3 + \frac{1}{2}\sum_{i=1}^{N}(-1)^i h_i \langle \partial_{\sigma_i} f_\varepsilon^\sigma w | w \rangle.$$

We use once again the IMS formula. We introduce the system of cut off functions $\chi_0, \dots, \chi_N$ satisfying (3.25) used for the proof of Proposition 3.4.
On the one hand we have:

$$I = -\langle \frac{h}{\varepsilon} \sin\varphi_\varepsilon^\sigma w | \mathcal{H}^\varepsilon(w) \rangle + \frac{1}{2}\sum_{i=1}^{N}(-1)^i h_i \langle \partial_{\sigma_i} f_\varepsilon^\sigma w | w \rangle.$$

Now, we remark that

$$\begin{aligned}
-\langle \frac{h}{\varepsilon} \sin\varphi_\varepsilon^\sigma w | \mathcal{H}^\varepsilon(w) \rangle = & -\sum_{i=1}^{N} \langle \frac{h}{\varepsilon} \sin\varphi_\varepsilon^\sigma \chi_i w | \chi_i \mathcal{H}^\varepsilon(w) \rangle \\
= & -\sum_{i=1}^{N} \langle \frac{h}{\varepsilon} \sin\varphi_\varepsilon^\sigma \chi_i w | \mathcal{H}^\varepsilon(\chi_i w) \rangle \\
& + \sum_{i=1}^{N} \langle \frac{h}{\varepsilon} \sin\varphi_\varepsilon^\sigma \chi_i w | \varepsilon \partial_{xx}(\chi_i w) - \varepsilon \chi_i \partial_{xx} w \rangle \\
= & -\sum_{i=1}^{N} \langle \frac{h}{\varepsilon} \sin\varphi_\varepsilon^\sigma \chi_i w | \mathcal{H}^\varepsilon(\chi_i w) \rangle \\
& + \sum_{i=1}^{N} \langle h \sin\varphi_\varepsilon^\sigma \chi_i w | 2\partial_x \chi_i \partial_x w + \partial_{xx}\chi_i w \rangle \\
= & -\sum_{i=1}^{N} \langle \frac{h}{\varepsilon} \sin\varphi_\varepsilon^\sigma \chi_i w | \mathcal{H}^\varepsilon(\chi_i w) \rangle - \langle h \sin\varphi_\varepsilon^\sigma (\sum_{i=0}^{N} (\partial_x \chi_i)^2) w | w \rangle
\end{aligned}$$

by differentiating the relation $\sum_{i=0}^{N}(\chi_i)^2 = 1$. On the other hand, we have

$$\langle \partial_{\sigma_i} f_\varepsilon^\sigma w | w \rangle = \langle \partial_{\sigma_i} f_\varepsilon^\sigma \chi_i w | \chi_i w \rangle + \langle \partial_{\sigma_i} f_\varepsilon^\sigma \chi_0 w | \chi_0 w \rangle$$

since the support of $\partial_{\sigma_i} f_\varepsilon^\sigma$ is contained in $[\sigma_i - \delta, \sigma_i + \delta]$. Therefore

$$I = I_0 + \sum_{i=1}^{N} I_{1,i} + I_2,$$

with

- $I_0 = -\langle \frac{h}{\varepsilon} \sin\varphi_\varepsilon^\sigma \chi_0 w | \mathcal{H}^\varepsilon(\chi_0 w) \rangle$,
- $I_{1,i} = -\langle \frac{h}{\varepsilon} \sin\varphi_\varepsilon^\sigma \chi_i w | \mathcal{H}^\varepsilon(\chi_i w) \rangle + (-1)^i \frac{h_i}{2} \langle \partial_{\sigma_i} f_\varepsilon^\sigma \chi_i w | \chi_i w \rangle$,
- $I_2 = -\langle h \sin\varphi_\varepsilon^\sigma (\sum_{i=0}^{N} (\partial_x \chi_i)^2) w | w \rangle + \sum_{i=1}^{N}(-1)^i h_i \langle \partial_{\sigma_i} f_\varepsilon^\sigma \chi_0 w | \chi_0 w \rangle$.

Third Step: Estimate in the domains

Concerning I_0, we have:

$$|I_0| \leq \frac{\|h\|_{L^\infty}}{\varepsilon} \|\mathbf{m}_\varepsilon \times \chi_0 w\|_{L^2} \|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(\chi_0 w)\|_{L^2}$$

(since $\mathbf{m}_\varepsilon \times (\mathbf{m}_\varepsilon \times w) = -w$). Now on the support of χ_0, $\mathbf{m}_\varepsilon = \pm e_1 + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$, and $f_\varepsilon^\sigma \geq \frac{1}{\varepsilon} - \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$. We denote by $\mathbf{v} = \chi_0 w$. From the relation $w \cdot \mathbf{m}_\varepsilon = 0$, we deduce that:

$$0 = \mathbf{v} \cdot \mathbf{m}_\varepsilon = \pm \mathbf{v}_1 + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}) \cdot \mathbf{v},$$

so

$$|\mathbf{v}_1| \leq \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})|\mathbf{v}|.$$

Therefore,

$$\begin{aligned}
\langle \mathbf{m}_\varepsilon \times \mathbf{v} | \mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(\mathbf{v}) \rangle = & \ \langle \mathbf{v} | \mathcal{H}^\varepsilon(\mathbf{v}) \rangle \\
= & \ \varepsilon \int_{[0,L]} |\partial_x \mathbf{v}|^2 - \frac{1}{\varepsilon} \int_{[0,L]} |v_1|^2 + \int [0,L] f_\varepsilon^\sigma |v|^2 \\
\geq & \ \frac{1}{\varepsilon} \|\mathbf{v}\|_{L^2}^2 - \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}) \|v\|_{L^2}^2.
\end{aligned}$$

So, with the Cauchy Schwartz inequality, we obtain that:

$$\|\mathbf{m}_\varepsilon \times \chi_0 w\|_{L^2} \leq (1 + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}))\varepsilon \|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(\chi_0 w)\|_{L^2}$$

Hence

$$|I_0| \leq (1 + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}))\|h\|_{L^\infty} \|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}^2. \tag{3.39}$$

Fourth Step: Estimate in the walls

Concerning $I_{1,i}$, on the support of χ_i, h is constant (by assumption (1.14)), $\mathbf{m}_\varepsilon$ writes:

$$\mathbf{m}_\varepsilon(x) = (-1)^{i+1} \mathbf{R}_{\frac{\theta_i}{\varepsilon}} M_0(\frac{x - \sigma_i}{\varepsilon}).$$

In addition, in this zone,

- $\sin \varphi_\varepsilon^\sigma = (-1)^{i+1} \tanh(\frac{x - \sigma_i}{\varepsilon})$
- $f_\varepsilon^\sigma(x) = \frac{1}{\varepsilon} f(\frac{x - \sigma_i}{\varepsilon})$, with $f(z) = 2\tanh^2 z - 1$,
- $\partial_{\sigma_i} f_\varepsilon^\sigma = -\frac{1}{\varepsilon^2} \partial_z f(\frac{x - \sigma_i}{\varepsilon})$.

We use the same decomposition of $\chi_i w$ as in (3.29):

$$(\chi_i w)(x) = r_1(\frac{x - \sigma_i}{\varepsilon}) \mathbf{R}_{\frac{\theta_i}{\varepsilon}} M_1(\frac{x - \sigma_i}{\varepsilon}) + r_2(\frac{x - \sigma_i}{\varepsilon}) \mathbf{R}_{\frac{\theta_i}{\varepsilon}} M_2.$$

As in Section 3.2, we obtain by (3.32) that in these new coordinates,

$$\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(\chi_j w)\|_{L^2}^2 = \frac{1}{\varepsilon} \left(\|\mathcal{L} r_1\|_{L^2}^2 + \|\mathcal{L} r_2\|_{L^2}^2 \right).$$

In addition, using the previous remarks and setting $z = \dfrac{x-\sigma_i}{\varepsilon}$ in the integrals, we obtain

$$I_{1,i} = h_i \frac{(-1)^{i+1}}{2} \int_{I\!R} \frac{1}{\varepsilon}\partial_z f r \cdot r dz + h_i \frac{(-1)^i}{\varepsilon} \int_{I\!R} \mathcal{L}r \cdot r \tanh z dz.$$

We remark that

$$\frac{1}{2}\int_{I\!R} \partial_z f r \cdot r dz = -\int_{I\!R} f r \cdot \partial_z r dz = -\int_{I\!R} \mathcal{L}r \cdot \partial_z r dz$$

since, the support of r being compact, $\int_{I\!R} \partial_{zz} r \cdot \partial_z r = 0$, therefore we obtain:

$$I_{1,i} = h_i \frac{(-1)^i}{\varepsilon} \int_{I\!R} \mathcal{L}r \cdot lr \, dr,$$

with $l = \partial_z + \tanh z$. We recall that from Proposition 3.5, we have:

$$\text{for } r \in (\frac{1}{\cosh z})^{\perp}, \ \|l(r)\|_{L^2} \leq \|\mathcal{L}r\|_{L^2}.$$

So

$$|I_{1,i}| \leq |h_i| \frac{1}{\varepsilon} \|\mathcal{L}r\|^2_{L^2},$$

that is

$$|I_{1,i}| \leq \|h\|_{L^\infty} \|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(\chi_i w)\|^2_{L^2}. \tag{3.40}$$

Fifth Step: end of the proof.

The last term I_2 is small. Indeed, on the support of χ_0, $|\partial_{\sigma_i} f^\sigma_\varepsilon| \leq \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$, so

$$|I_2| \leq c\|h\|_{L^\infty}\|w\|^2_{L^2} + \|h\|_{L^\infty}\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})\|w\|^2_{L^2} \leq c\varepsilon^2 \|w \times \mathcal{H}^\varepsilon(w)\|^2_{L^2} \tag{3.41}$$

with Proposition 3.4.

The last remaining term to estimate is $\langle l^\varepsilon w | \mathcal{H}^\varepsilon(w)\rangle$. Since $l^\varepsilon = \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$, we obtain that

$$\left|\langle l^\varepsilon w | \mathcal{H}^\varepsilon(w)\rangle\right| \leq \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})\|w \times \mathcal{H}^\varepsilon(w)\|^2_{L^2}. \tag{3.42}$$

So adding up the previous estimates (3.37), (3.38), (3.39), (3.40), (3.41) and (3.42), we obtain that

$$|I| \leq (1 + c\varepsilon^2)\|h\|_{L^\infty} \sum_{i=0}^{N} \|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|^2_{L^2}. \tag{3.43}$$

We observe that

$$\mathcal{H}^\varepsilon(\chi_i w) = \chi_i \mathcal{H}^\varepsilon(w) + \tau_i,$$

with $\tau_i = -\varepsilon(2\partial_x \chi_i \partial_x w + \partial_{xx}\chi_i w$. This term is small since:

$$\|\tau_i\|_{L^2} \leq c\varepsilon \|\chi_i\|_{L^\infty}\|\partial_x w\|_{L^2} + c\varepsilon\|\partial_{xx}\chi_i\|_{L^2}\|w\|_{L^2} \leq c\sqrt{\varepsilon}\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}$$

by Proposition 3.2. So

$$\sum_{i=0}^{N} \|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|^2_{L^2} = \|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|^2_{L^2} + 2\sum_i \|\tau_i\|_{L^2}\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2} + \sum_i \|\tau_i\|_{L^2},$$

thus we obtain that

$$\sum_{i=0}^{N} \|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|^2_{L^2} \leq (1 + c\sqrt{\varepsilon})\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|^2_{L^2}.$$

Together with (3.43) we obtain that

$$|I| \leq \|h\|_{L^\infty}(1 + c\sqrt{\varepsilon})\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|^2_{L^2}. \tag{3.44}$$

This conclude the proof of Proposition 3.3. □

4 Equations in the new coordinates

From Proposition 1.1, we know that, while the solution m of the Landau-Lifschitz equation (1.4) remains in a neighborhood of $\mathcal{M}_\delta^\varepsilon$, we can write m on the form

$$m(t,x) = \mathbf{m}_\varepsilon(\theta(t),\sigma(t)) + w(t) + \nu(w(t))\mathbf{m}_\varepsilon(\theta(t),\sigma(t)).$$

We plug this formula in (1.4). Concerning the time derivatives, we have:

$$\frac{d}{dt}\left[t\mapsto \mathbf{m}_\varepsilon(\theta(t),\sigma(t))\right] = \sum_{i=1}^N \left(\partial_{\theta_i}\mathbf{m}_\varepsilon(\theta,\sigma)\frac{d\theta_i}{dt} + \partial_{\sigma_i}\mathbf{m}_\varepsilon(\theta,\sigma)\frac{d\sigma_i}{dt}\right)$$

and

$$\frac{d}{dt}\left[\nu(w(t))\right] = \nu'(w(t))(\frac{dw}{dt}(t)).$$

Using Proposition 2.1, we have:

$$\begin{aligned} h^\varepsilon(m) = \quad & f_\varepsilon^\sigma \mathbf{m}_\varepsilon(\theta,\sigma) + \rho_\varepsilon^{\theta,\sigma} + h^\varepsilon(w) + \nu(w)(f_\varepsilon^\sigma\mathbf{m}_\varepsilon(\theta,\sigma) + \rho_\varepsilon^{\theta,\sigma}) + 2\varepsilon v'(w)(\partial_x w)\partial_x\mathbf{m}_\varepsilon \\ & + \varepsilon(\nu'(w)\partial_{xx}w + \nu''(w)(\partial_x w,\partial_x w))\mathbf{m}_\varepsilon. \end{aligned}$$

In addition, in the i^{th} wall, *i.e.* in $[\sigma_i-\delta,\sigma_i+\delta]$,

$$\mathbf{m}_\varepsilon(\theta,\sigma)(x) = \mathbf{R}_{\frac{\theta_i}{\varepsilon}}\begin{pmatrix}(-1)^{i+1}\sin\varphi_\varepsilon(x-\sigma_i)\\ \cos\varphi_\varepsilon(x-\sigma_i)\\ 0\end{pmatrix},$$

so,

$$-\frac{1}{\varepsilon}\mathbf{m}_\varepsilon\times e_1 = \partial_{\theta_i}\mathbf{m}_\varepsilon,$$

and

$$\begin{aligned} -\frac{1}{\varepsilon}\mathbf{m}_\varepsilon\times(\mathbf{m}_\varepsilon\times e_1) = \quad & \frac{1}{\varepsilon}\cos\varphi_\varepsilon(x-\sigma_i)\mathbf{R}_{\frac{\theta_i}{\varepsilon}}\begin{pmatrix}\cos\varphi_\varepsilon(x-\sigma_i)\\ (-1)^i\sin\varphi_\varepsilon(x-\sigma_i)\\ 0\end{pmatrix} \\ = \quad & \left[\frac{d\varphi_\varepsilon}{dx}(x-\sigma_i) - \beta_\varepsilon(x-\sigma_i)\right]\mathbf{R}_{\frac{\theta_i}{\varepsilon}}\begin{pmatrix}\cos\varphi_\varepsilon(x-\sigma_i)\\ (-1)^i\sin\varphi_\varepsilon(x-\sigma_i)\\ 0\end{pmatrix} \\ = \quad & (-1)^i\partial_{\sigma_i}\mathbf{m}_\varepsilon(\theta,\sigma)(x) + q_\varepsilon^{\theta,\sigma}, \end{aligned}$$

where

$$q_\varepsilon^{\theta,\sigma} = -\beta_\varepsilon(x-\sigma_i)\mathbf{R}_{\frac{\theta_i}{\varepsilon}}\begin{pmatrix}\cos\varphi_\varepsilon(x-\sigma_i)\\ (-1)^i\sin\varphi_\varepsilon(x-\sigma_i)\\ 0\end{pmatrix} = \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$$

(see Lemma 2.1).
So we obtain that

$$\begin{aligned} \sum_{i=1}^N(1+\nu(w))\partial_{\theta_i}\mathbf{m}_\varepsilon\frac{d\theta_i}{dt} + \sum_{i=1}^N(1+\nu(w))\partial_{\sigma_i}\mathbf{m}_\varepsilon\frac{d\sigma_i}{dt} + \frac{\partial w}{\partial t} + \nu'(w)(\frac{\partial w}{\partial t})\mathbf{m}_\varepsilon = \\ T_1+\ldots+T_6, \end{aligned} \tag{4.1}$$

with

- $T_1 = -\mathbf{m}_\varepsilon\times\rho_\varepsilon^{\theta,\sigma} - \mathbf{m}_\varepsilon\times(\mathbf{m}_\varepsilon\times\rho_\varepsilon^{\theta,\sigma}) = \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$,

- $T_2 = -\frac{h}{\varepsilon}\mathbf{m}_\varepsilon \times e_1 - \frac{h}{\varepsilon}\mathbf{m}_\varepsilon \times (\mathbf{m}_\varepsilon \times e_1) = h\sum_{i=1}^{N}(\partial_{\theta_i}\mathbf{m}_\varepsilon + (-1)^i\partial_{\sigma_i}\mathbf{m}_\varepsilon) + hq_\varepsilon^{\theta,\sigma}$,
- $T_3 = -\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w) - \mathbf{m}_\varepsilon \times (\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w))$, where

$$\mathcal{H}^\varepsilon(w) = -h^\varepsilon(w) + f_\varepsilon^\sigma w = -\varepsilon\partial_{xx}w - \frac{1}{\varepsilon}w_1e_1 + f_\varepsilon^\sigma w,$$

- $T_4 = -\frac{h}{\varepsilon}w \times e_1 - \frac{h}{\varepsilon}(\sin\varphi_\varepsilon^\sigma w + w_1\mathbf{m}_\varepsilon)$,
- $T_5 = -w \times \rho_\varepsilon^{\theta,\sigma} - \mathbf{m}_\varepsilon \times (w \times \rho_\varepsilon^{\theta,\sigma}) - w \times (\mathbf{m}_\varepsilon \times \rho_\varepsilon^{\theta,\sigma})$. This term is linear in w but exponentially small:

$$T_5 = \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})(w).$$

- T_6 is the non linear part for the variable w. We split it into 4 terms: $T_6 = T_{61} + \ldots + T_{64}$ with

$$\begin{aligned}
T_{61} = \quad & -\varepsilon(w + \nu(w)\mathbf{m}_\varepsilon) \times (\partial_{xx}w + \nu'(w)(\partial_{xx}w)\mathbf{m}_\varepsilon) \\
& -\varepsilon(\mathbf{m}_\varepsilon + w + \nu(w)\mathbf{m}_\varepsilon) \times ((w + \nu(w)\mathbf{m}_\varepsilon) \times (\partial_{xx}w + \nu'(w)(\partial_{xx}w)\mathbf{m}_\varepsilon)) \\
& -\varepsilon(w + \nu(w)\mathbf{m}_\varepsilon) \times (\mathbf{m}_\varepsilon \times (\partial_{xx}w + \nu'(w)(\partial_{xx}w)\mathbf{m}_\varepsilon)), \\
T_{62} = \quad & -\varepsilon\nu''(w)(\partial_x w, \partial_x w)(w \times \mathbf{m}_\varepsilon + (\mathbf{m}_\varepsilon + w + \nu(w)\mathbf{m}_\varepsilon) \times (w \times \mathbf{m}_\varepsilon)), \\
T_{63} = \quad & -2\varepsilon\nu'(w)(\partial_x w)\left((\mathbf{m}_\varepsilon + w + \nu(w)\mathbf{m}_\varepsilon) \times \partial_x\mathbf{m}_\varepsilon\right. \\
& \left. + (\mathbf{m}_\varepsilon + w + \nu(w)\mathbf{m}_\varepsilon) \times ((\mathbf{m}_\varepsilon + w + \nu(w)\mathbf{m}_\varepsilon) \times \partial_x\mathbf{m}_\varepsilon)\right), \\
T_{64} = \quad & -\nu(w)\mathbf{m}_\varepsilon \times \left(\frac{h}{\varepsilon}e_1 + \rho_\varepsilon^{\theta,\sigma} + \nu(w)\rho_\varepsilon^{\theta,\sigma} + \frac{1}{\varepsilon}w_1e_1\right) - w \times \left(\frac{1}{\varepsilon}w_1e_1 + \nu(w)(f_\varepsilon^\sigma\mathbf{m}_\varepsilon + \rho_\varepsilon^{\theta,\sigma})\right) \\
& -\mathbf{m}_\varepsilon \times (\nu(w)\rho_\varepsilon^{\theta,\sigma}) - (\mathbf{m}_\varepsilon + w + \nu(w)\mathbf{m}_\varepsilon) \times \left[-w \times \left(\frac{1}{\varepsilon}w_1e_1 + \nu(w)(f_\varepsilon^\sigma\mathbf{m}_\varepsilon + \rho_\varepsilon^{\theta,\sigma})\right)\right. \\
& \left. -\nu(w)\mathbf{m}_\varepsilon \times \left(\frac{h}{\varepsilon}e_1 + \rho_\varepsilon^{\theta,\sigma} + \nu(w)\rho_\varepsilon^{\theta,\sigma} + \frac{1}{\varepsilon}w_1e_1\right) - \mathbf{m}_\varepsilon \times (\nu(w)\rho_\varepsilon^{\theta,\sigma})\right] \\
& -(w + \nu(w)\mathbf{m}_\varepsilon) \times \left(w \times (f_\varepsilon^\sigma\mathbf{m}_\varepsilon + \frac{h}{\varepsilon}\mathbf{m}_\varepsilon e_1 + \rho_\varepsilon^{\theta,\sigma}) + \mathbf{m}_\varepsilon \times \frac{1}{\varepsilon}w_1e_1\right) \\
& -\nu(w)\mathbf{m}_\varepsilon \times (\mathbf{m}_\varepsilon \times (\frac{h}{\varepsilon}e_1 + \rho_\varepsilon^{\theta,\sigma})).
\end{aligned} \tag{4.2}$$

Therefore there exists K independent of ε such that

$$|T_{61}| \leq K\varepsilon|w||\partial_{xx}w|, \ \ |T_{62}| \leq K\varepsilon|\partial_x w|^2|w|, \ |T_{63}| \leq K|w||\partial_x w|, \ \ |T_{64}| \leq \frac{K}{\varepsilon}|w|^2. \tag{4.3}$$

In order to isolate the equation in each variable, we take the L^2 scalar product of the previous equation with $\partial_{\theta_i}\mathbf{m}_\varepsilon$ and with $\partial_{\sigma_i}\mathbf{m}_\varepsilon$. In this section, we assume that the applied field h is constant with respect to x in the walls $[\sigma_i - \delta, \sigma_i + \delta]$. From Assumption 1.14, this condition is realized while $|\sigma_i(t) - \sigma_i^{ref}(t)| \leq \delta$ for all $i \in \{1, \ldots, N\}$.

4.1 Equation for $\frac{d\theta_i}{dt}$ and $\frac{d\sigma_i}{dt}$

We first take the L^2 inner product of (4.1) with $\partial_{\theta_i}\mathbf{m}_\varepsilon$. We remark that $\partial_{\theta_i}\mathbf{m}_\varepsilon \cdot \mathbf{m}_\varepsilon = 0$, $\partial_{\theta_i}\mathbf{m}_\varepsilon \cdot \partial_{\sigma_i}\mathbf{m}_\varepsilon = 0$ and that $\langle w|\partial_{\theta_i}\mathbf{m}_\varepsilon\rangle = 0$. In addition, we have

1. $\langle \partial_{\theta_i}\mathbf{m}_\varepsilon | \partial_{\theta_i}\mathbf{m}_\varepsilon \rangle = \dfrac{2}{\varepsilon} + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$.

2. $\langle \mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w) + \mathbf{m}_\varepsilon \times (\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)) | \partial_{\theta_i}\mathbf{m}_\varepsilon \rangle = \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$. Indeed,

$$\begin{aligned} \langle \mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w) | \partial_{\theta_i}\mathbf{m}_\varepsilon \rangle &= -\langle \mathbf{m}_\varepsilon \times \partial_{\theta_i}\mathbf{m}_\varepsilon | \mathcal{H}^\varepsilon(w) \rangle \\ &= (-1)^{i+1}\langle \partial_{\sigma_i}\mathbf{m}_\varepsilon | \mathcal{H}^\varepsilon(w) \rangle + \langle \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}) | \mathcal{H}^\varepsilon(w) \rangle \\ &= (-1)^{i+1}\langle \mathcal{H}^\varepsilon(\partial_{\sigma_i}\mathbf{m}_\varepsilon) | w \rangle + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})(w) \\ &\quad \text{by integrations by parts.} \end{aligned}$$

Now we have the following proposition:

Proposition 4.1. *For $\theta \in I\!R^N$, for $\sigma \in \Sigma_\delta$, on $[-L, L]$,*

$$h^\varepsilon(\partial_{\sigma_i}\mathbf{m}_\varepsilon(\theta,\sigma)) = f_\varepsilon^\sigma(x)\partial_{\sigma_i}\mathbf{m}_\varepsilon(\theta,\sigma) + \partial_{\sigma_i}f_\varepsilon^\sigma \mathbf{m}_\varepsilon(\theta,\sigma) + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}),$$

and

$$h^\varepsilon(\partial_{\theta_i}\mathbf{m}_\varepsilon(\theta,\sigma)) = f_\varepsilon^\sigma(x)\partial_{\theta_i}\mathbf{m}_\varepsilon(\theta,\sigma) + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}).$$

Proof. By taking the derivative of the expression of $h^\varepsilon(\mathbf{m}_\varepsilon(\theta,\sigma))$ with respect to σ_i and θ_i (see Proposition 2.1), we obtain directly the claimed results. ☐

So, $\mathcal{H}^\varepsilon(\partial_{\sigma_i}\mathbf{m}_\varepsilon) = h^\varepsilon(\partial_{\sigma_i}\mathbf{m}_\varepsilon) - f_\varepsilon^\sigma \partial_{\sigma_i}\mathbf{m}_\varepsilon = \partial_{\sigma_i}f_\varepsilon^\sigma \mathbf{m}_\varepsilon(\theta,\sigma) + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$, and since $\mathbf{m}_\varepsilon \cdot w = 0$, we obtain that

$$\langle \mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w) | \partial_{\theta_i}\mathbf{m}_\varepsilon \rangle = \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})(w)$$

In the same way,

$$\begin{aligned} \langle \mathbf{m}_\varepsilon \times (\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)) | \partial_{\theta_i}\mathbf{m}_\varepsilon \rangle &= -\langle \mathcal{H}^\varepsilon(w) | \partial_{\theta_i}\mathbf{m}_\varepsilon \rangle \\ &\quad \text{since } \mathbf{m}_\varepsilon \cdot \partial_{\theta_i}\mathbf{m}_\varepsilon = 0 \text{ and since } |\mathbf{m}_\varepsilon| = 1 \\ &= -\langle w | \mathcal{H}^\varepsilon(\partial_{\theta_i}\mathbf{m}_\varepsilon) \rangle \\ &= \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})(w) \text{ by Proposition 4.1.} \end{aligned}$$

3. For all t, $\langle w | \partial_{\theta_i}\mathbf{m}_\varepsilon \rangle = 0$. We differentiate this equality with respect to t and we obtain:

$$\begin{aligned} \langle \frac{\partial w}{\partial t} | \partial_{\theta_i}\mathbf{m}_\varepsilon \rangle &= -\langle w | \frac{\partial}{\partial t}(\partial_{\theta_i}\mathbf{m}_\varepsilon) \rangle \\ &= -\langle w | \partial_{\theta_i}\partial_{\theta_i}\mathbf{m}_\varepsilon \rangle \frac{d\theta_i}{dt} - \langle w | \partial_{\sigma_i}\partial_{\theta_i}\mathbf{m}_\varepsilon \rangle \frac{d\sigma_i}{dt}, \end{aligned}$$

and by (2.7), we get:

$$\langle \frac{\partial w}{\partial t} | \partial_{\theta_i}\mathbf{m}_\varepsilon \rangle = \left(\frac{(-1)^{i+1}}{\varepsilon}\Pi_\varepsilon^{\sigma_i}(w) + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})(w) \right) \frac{d\theta_i}{dt} + \left(\frac{(-1)^i}{\varepsilon}\Pi_\varepsilon^{\theta_i}(w) + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})(w) \right) \frac{d\sigma_i}{dt},$$

where

$$\Pi_\varepsilon^{\theta_i}(w) = \langle \sin\varphi_\varepsilon^\sigma \partial_{\theta_i}\mathbf{m}_\varepsilon | w \rangle = \int_{\sigma_i-\delta}^{\sigma_i+\delta} \sin\varphi_\varepsilon^\sigma w \cdot \partial_{\theta_i}\mathbf{m}_\varepsilon,$$

$$\Pi_\varepsilon^{\sigma_i}(w) = \langle \sin\varphi_\varepsilon^\sigma \partial_{\sigma_i}\mathbf{m}_\varepsilon | w \rangle = \int_{\sigma_i-\delta}^{\sigma_i+\delta} \sin\varphi_\varepsilon^\sigma w \cdot \partial_{\sigma_i}\mathbf{m}_\varepsilon.$$

4. We assume in this section that the applied field $h(t,x)$ is constant equal to h_i in the wall i, so that we have:

$$\begin{aligned}\big\langle T_2|\partial_{\theta_i}\mathbf{m}_\varepsilon\big\rangle = & \big\langle h\partial_{\theta_i}\mathbf{m}_\varepsilon + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})|\partial_{\theta_i}\mathbf{m}_\varepsilon\big\rangle \\ = & \frac{2h_i}{\varepsilon} + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}).\end{aligned}$$

In addition,

$$\begin{aligned}\big\langle T_4|\partial_{\theta_i}\mathbf{m}_\varepsilon\big\rangle = & \frac{h_i}{\varepsilon}\left(-\big\langle w\times e_1|\partial_{\theta_i}\mathbf{m}_\varepsilon\big\rangle - \big\langle \sin\varphi_\varepsilon^\sigma w|\partial_{\theta_i}\mathbf{m}_\varepsilon\big\rangle\right) \\ = & \frac{h_i}{\varepsilon}\left(\big\langle e_1\times\partial_{\theta_i}\mathbf{m}_\varepsilon|w\big\rangle - \big\langle \sin\varphi_\varepsilon^\sigma w|\partial_{\theta_i}\mathbf{m}_\varepsilon\big\rangle\right) \\ = & h_i\left((-1)^{i+1}\frac{1}{\varepsilon}\Pi_\varepsilon^{\sigma_i}(w) - \frac{1}{\varepsilon}\Pi_\varepsilon^{\theta_i}(w)\right) + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})(w).\end{aligned}$$

So by projection of (4.1) on $\partial_{\theta_i}\mathbf{m}_\varepsilon$, we obtain that

$$\begin{aligned}&\left(\frac{2}{\varepsilon} + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}) + (-1)^{i+1}\frac{1}{\varepsilon}\Pi_\varepsilon^{\sigma_i}(w) + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})(w) + \frac{1}{\varepsilon}Q_\varepsilon^{\theta_i}(w)\right)\frac{d\theta_i}{dt} \\ &+ \left((-1)^i\frac{1}{\varepsilon}\Pi_\varepsilon^{\theta_i}(w) + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})(w)\right)\frac{d\sigma_i}{dt} = \qquad (4.4) \\ &\frac{2h_i}{\varepsilon} + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}) + (-1)^{i+1}h_i\frac{1}{\varepsilon}\Pi_\varepsilon^{\sigma_i}(w) - h_i\frac{1}{\varepsilon}\Pi_\varepsilon^{\theta_i}(w) + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})(w) + \big\langle T_6|\partial_{\theta_i}\mathbf{m}_\varepsilon\big\rangle,\end{aligned}$$

with

$$Q_\varepsilon^{\theta_i}(w) = \varepsilon\big\langle \nu(w)\partial_{\theta_i}\mathbf{m}_\varepsilon|\partial_{\theta_i}\mathbf{m}_\varepsilon\big\rangle = \varepsilon\int_{\sigma_i-\delta}^{\sigma_i+\delta}\nu(w)|\partial_{\theta_i}\mathbf{m}_\varepsilon|^2.$$

In the same way, we take the scalar product of (4.1) with respect to $\partial_{\sigma_i}\mathbf{m}_\varepsilon$. As before, and basically with the same arguments, we remark that:

1. $\big\langle\partial_{\sigma_i}\mathbf{m}_\varepsilon|\partial_{\sigma_i}\mathbf{m}_\varepsilon\big\rangle = \dfrac{2}{\varepsilon} + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$,
2. $\big\langle\mathbf{m}_\varepsilon\times\mathcal{H}^\varepsilon(w) + \mathbf{m}_\varepsilon\times(\mathbf{m}_\varepsilon\times\mathcal{H}^\varepsilon(w))|\partial_{\sigma_i}\mathbf{m}_\varepsilon\big\rangle = \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$.
3. Taking the derivative with respect to t of the relation $\big\langle w|\partial_{\sigma_i}\mathbf{m}_\varepsilon\big\rangle = 0$, we obtain that:

$$\begin{aligned}\Big\langle\frac{\partial w}{\partial t}|\partial_{\sigma_i}\mathbf{m}_\varepsilon\Big\rangle = & -\Big\langle w|\frac{\partial}{\partial t}(\partial_{\sigma_i}\mathbf{m}_\varepsilon)\Big\rangle \\ = & -\big\langle w|\partial_{\theta_i}\partial_{\sigma_i}\mathbf{m}_\varepsilon\big\rangle\frac{d\theta_i}{dt} - \big\langle w|\partial_{\sigma_i}\partial_{\sigma_i}\mathbf{m}_\varepsilon\big\rangle\frac{d\sigma_i}{dt} \\ = & \left(\frac{(-1)^i}{\varepsilon}\Pi_\varepsilon^{\sigma_i}(w) + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})(w)\right)\frac{d\sigma_i}{dt} + \left(\frac{(-1)^i}{\varepsilon}\Pi_\varepsilon^{\theta_i}(w) + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})(w)\right)\frac{d\theta_i}{dt}.\end{aligned}$$

4. The applied field $h(t,x)$ is constant and equal to h_i in the wall i, so we have:

$$\begin{aligned}\big\langle T_2|\partial_{\sigma_i}\mathbf{m}_\varepsilon\big\rangle = & (-1)^i\big\langle h\partial_{\sigma_i}\mathbf{m}_\varepsilon + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})|\partial_{\sigma_i}\mathbf{m}_\varepsilon\big\rangle \\ = & (-1)^i\frac{2h_i}{\varepsilon} + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}).\end{aligned}$$

In addition,

$$\begin{aligned}\langle T_4|\partial_{\sigma_i}\mathbf{m}_\varepsilon\rangle = & \frac{h_i}{\varepsilon}\left(-\langle w\times e_1|\partial_{\sigma_i}\mathbf{m}_\varepsilon\rangle - \langle \sin\varphi_\varepsilon^\sigma w|\partial_{\sigma_i}\mathbf{m}_\varepsilon\rangle\right)\\ = & h_i\left((-1)^i\frac{1}{\varepsilon}\Pi_\varepsilon^{\theta_i}(w)+\frac{1}{\varepsilon}\Pi_\varepsilon^{\sigma_i}(w)\right)+\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})(w).\end{aligned}$$

So we obtain

$$\begin{gathered}\left((-1)^i\frac{1}{\varepsilon}\Pi_\varepsilon^{\theta_i}(w)+\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})(w)\right)\frac{d\theta_i}{dt}\\ +\left(\frac{2}{\varepsilon}+\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})+(-1)^i\frac{1}{\varepsilon}\Pi_\varepsilon^{\sigma_i}(w)+\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})(w)+\frac{1}{\varepsilon}Q_\varepsilon^{\sigma_i}(w)\right)\frac{d\sigma_i}{dt}= \qquad (4.5)\\ (-1)^i\frac{2h_i}{\varepsilon}+\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})+(-1)^ih_i\frac{1}{\varepsilon}\Pi_\varepsilon^{\theta_i}(w)-h_i\frac{1}{\varepsilon}\Pi_\varepsilon^{\sigma_i}(w)+\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})(w)+\langle T_6|\partial_{\sigma_i}\mathbf{m}_\varepsilon\rangle,\end{gathered}$$

with

$$Q_\varepsilon^{\sigma_i}(w)=\varepsilon\langle\nu(w)\partial_{\sigma_i}\mathbf{m}_\varepsilon|\partial_{\sigma_i}\mathbf{m}_\varepsilon\rangle=\varepsilon\int_{\sigma_i-\delta}^{\sigma_i+\delta}\nu(w)|\partial_{\sigma_i}\mathbf{m}_\varepsilon|^2.$$

We rewrite both equations (4.4) and (4.5) in the condensed form:

$$(I+L^i(w)+Q^i(w)+\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}))\frac{dv_i}{dt}=\begin{pmatrix}h_i\\(-1)^ih_i\end{pmatrix}+\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})+h_iR^i(w)+F^i, \qquad (4.6)$$

where

$$v_i=\begin{pmatrix}\theta_i\\ \sigma_i\end{pmatrix},\quad I=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$

$$L^i(w)=\frac{(-1)^i}{2}\begin{pmatrix}-\Pi_\varepsilon^{\sigma_i}(w)&\Pi_\varepsilon^{\theta_i}(w)\\ \Pi_\varepsilon^{\theta_i}(w)&\Pi_\varepsilon^{\sigma_i}(w)\end{pmatrix}+\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})(w),$$

$$Q^i(w)=\begin{pmatrix}\frac{1}{2}Q_\varepsilon^{\theta_i}(w)&0\\0&\frac{1}{2}Q_\varepsilon^{\sigma_i}(w)\end{pmatrix},\quad R^i(w)=\begin{pmatrix}-\frac{1}{2}\Pi_\varepsilon^{\theta_i}(w)+\frac{(-1)^{i+1}}{2}\Pi_\varepsilon^{\sigma_i}(w)\\ -\frac{1}{2}\Pi_\varepsilon^{\sigma_i}(w)+\frac{(-1)^i}{2}\Pi_\varepsilon^{\theta_i}(w)\end{pmatrix},$$

$$F^i=\begin{pmatrix}\langle T_6|\partial_{\theta_i}\mathbf{m}_\varepsilon\rangle\\ \langle T_6|\partial_{\sigma_i}\mathbf{m}_\varepsilon\rangle\end{pmatrix}.$$

We have the following estimates:

Proposition 4.2. *There exists a constant K independent of ε such that for all $\theta\in\mathbb{R}^N$ and for all $\sigma\in\Sigma_\delta$, for all $w\in H^1([0,L];\mathbb{R}^3)$,*

$$|\Pi_\varepsilon^{\sigma_i}(w)|+|\Pi_\varepsilon^{\theta_i}(w)|\le K\|w\|_{L^\infty}$$

and

$$|Q_\varepsilon^{\sigma_i}(w)|+|Q_\varepsilon^{\theta_i}(w)|\le K\|w\|^2_{L^\infty}.$$

Proof. On the one hand, we have:

$$\begin{aligned} |\Pi_\varepsilon^{\sigma_i}(w)| = & \left|\int_{\sigma_i-\delta}^{\sigma_i+\delta} \sin\varphi_\varepsilon^\sigma \partial_{\sigma_i}\mathbf{m}_\varepsilon \cdot w\right| \\ \leq & \ \|w\|_{L^\infty}\|\partial_{\sigma_i}\mathbf{m}_\varepsilon\|_{L^1} \\ \leq & \ K\|w\|_{L^\infty} \text{ by Proposition 2.2.} \end{aligned}$$

We estimate $\Pi_\varepsilon^{\theta_i}(w)$ with the same arguments.

On the other hand,

$$\begin{aligned} |Q_\varepsilon^{\sigma_i}(w)| = & \ \varepsilon\int_{\sigma_i-\delta}^{\sigma_i+\delta} \nu(w)|\partial_{\sigma_i}\mathbf{m}_\varepsilon|^2 \\ \leq & \ \|\nu(w)\|_{L^\infty}\left(2+\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})\right) \\ \leq & \ K\|w\|_{L^\infty}^2. \end{aligned}$$

The same holds for $Q_\varepsilon^{\theta_i}(w)$.

□

Proposition 4.3. *There exists $\nu_0 > 0$ such that if $\|w\|_{L^\infty} \leq \nu_0$, then for all ε small enough, the matrix $I + L^i(w) + Q^i(w) + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$ in Equation (4.6) is invertible and*

$$\left(I + L^i(w) + Q^i(w) + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})\right)^{-1} = I - L^i(w) + \tilde{Q}^i(w) + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}),$$

where

$$\left|\tilde{Q}^i(w)\right| \leq C\|w\|_{L^\infty}^2.$$

Proof. From Proposition 4.2, for ε small enough, if $\|w\|_{L^\infty}$ is small enough then $L^i(w) + Q^i(w) + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$ is a small perturbation of I, so $I + L^i(w) + Q^i(w) + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$ is invertible and writing the inverse on the form of a power series, we conclude the proof of Proposition 4.3.

From (4.6) and Proposition 4.3 we obtain that:

$$\begin{cases} \dfrac{d\theta_i}{dt} = h_i + a_\varepsilon^{\theta_i} - h_i\Pi_\varepsilon^{\theta_i}(w) + l_\varepsilon^{\theta_i}(w) + G_\varepsilon^{\theta_i}(\theta_i,\sigma_i,w) \\ \dfrac{d\sigma_i}{dt} = (-1)^i h_i + a_\varepsilon^{\sigma_i} - h_i\Pi_\varepsilon^{\sigma_i}(w) + l_\varepsilon^{\sigma_i}(w) + G_\varepsilon^{\sigma_i}(\theta_i,\sigma_i,w) \end{cases} \tag{4.7}$$

where

$$\begin{pmatrix} G^{\theta_i}(\theta_i,\sigma_i,w) \\ G^{\sigma_i}(\theta_i,\sigma_i,w)\end{pmatrix} = \left(I - L^i w + \tilde{Q}_i(w) + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})\right)\frac{\varepsilon}{2}\begin{pmatrix}\langle T_6|\partial_{\theta_i}\mathbf{m}_\varepsilon\rangle \\ \langle T_6|\partial_{\sigma_i}\mathbf{m}_\varepsilon\rangle\end{pmatrix}$$

$$+h_i\left(-L^i w + \tilde{Q}^i(w)\right)R^i w + h_i\tilde{Q}^i(w)\begin{pmatrix}1 \\ (-1)^i\end{pmatrix},$$

and where $a_\varepsilon^{\theta_i}$, $a_\varepsilon^{\sigma_i}$, $l_\varepsilon^{\theta_i}$ and $l_\varepsilon^{\sigma_i}$ of order $\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$.

□

The non linear terms $G_\varepsilon^{\theta_i}$ and $G_\varepsilon^{\sigma_i}$ can be estimated as follows:

Proposition 4.4. *There exists a constant K independent of $\varepsilon > 0$, such that for all θ and all $\sigma \in \Sigma_\delta$,*

$$|G_\varepsilon^{\theta_i}(\theta_i,\sigma_i,w)| + |G_\varepsilon^{\sigma_i}(\theta_i,\sigma_i,w)| \leq K\left(\frac{1}{\varepsilon}\|w\|_{L^2}^2 + \varepsilon\|\partial_x w\|_{L^2}^2\right) \leq K\|w\|_\varepsilon^2.$$

Proof. On the one hand, from Proposition 4.2, we have:

$$|L^i w| \leq K\|w\|_{L^\infty} \text{ and } |R^i w| \leq K\|w\|_{L^\infty}.$$

In addition, with Proposition 4.3,

$$|\tilde{Q}^i(w)| \leq K\|w\|_{L^\infty}^2.$$

Let us estimate $\varepsilon\langle T_6|\partial_{\theta_i}\mathbf{m}_\varepsilon\rangle$. We recall that $T_6 = T_{61} + \ldots + T_{64}$ (see (4.2) at the beginning of Section 4).

The first term T_{61} writes $T_{61} = \varepsilon g(x,\varepsilon,w)(\partial_{xx}w)$, where $g : [0,L]\times]0,1[\times I\!R^3 \to \mathcal{L}(I\!R^3; I\!R^3)$ is smooth in its variables, with

$$|g| \leq K|w|, \quad |\partial_x g| \leq K|\partial_x\mathbf{m}_\varepsilon|\,|w| \quad \text{and } |\partial_w g| \leq K.$$

We have:

$$\begin{aligned}
\big\langle T_{61}|\partial_{\theta_i}\mathbf{m}_\varepsilon\big\rangle &= \varepsilon\big\langle g(x,\varepsilon,w)\partial_{xx}w|\partial_{\theta_i}\mathbf{m}_\varepsilon\big\rangle\\
&= \varepsilon\big\langle \partial_{xx}w|{}^\tau g(x,\varepsilon,w)\partial_{\theta_i}\mathbf{m}_\varepsilon\big\rangle\\
&= -\varepsilon\big\langle \partial_x w|\partial_x({}^\tau g(x,\varepsilon,w)\partial_{\theta_i}\mathbf{m}_\varepsilon)\big\rangle\\
&= \tau_1+\tau_2+\tau_3,
\end{aligned}$$

where each τ_i is defined and estimated in the following way, using that $\|\partial_x\mathbf{m}_\varepsilon\|_{L^\infty} + \|\partial_{\theta_i}\mathbf{m}_\varepsilon\|_{L^\infty} \leq \dfrac{K}{\varepsilon}$ and that $\|\partial_{\theta_i}\mathbf{m}_\varepsilon\|_{L^2} \leq \dfrac{K}{\sqrt{\varepsilon}}$ (see Proposition 2.2):

$$\begin{aligned}
\tau_1 = &\ -\varepsilon\big\langle \partial_x w|{}^\tau\partial_x g(x,\varepsilon,w)\partial_{\theta_i}\mathbf{m}_\varepsilon\big\rangle,\\
|\tau_1| \leq &\ K\varepsilon\|w\|_{L^\infty}\|\partial_x\mathbf{m}_\varepsilon\|_{L^\infty}\|\partial_x w\|_{L^2}\|\partial_{\theta_i}\mathbf{m}_\varepsilon\|_{L^2}\\
\leq &\ \frac{K}{\sqrt{\varepsilon}}\|w\|_{L^\infty}\|\partial_x w\|_{L^2},\\
\tau_2 = &\ -\varepsilon\big\langle \partial_x w|{}^\tau\partial_w g(x,\varepsilon,w)(\partial_x w)\partial_{\theta_i}\mathbf{m}_\varepsilon\big\rangle,\\
|\tau_2| \leq &\ K\varepsilon\|\partial_x w\|_{L^2}^2\|\partial_{\theta_i}\mathbf{m}_\varepsilon\|_{L^\infty}\\
\leq &\ K\|\partial_x w\|_{L^2}^2,\\
\tau_3 = &\ -\varepsilon\big\langle \partial_x w|{}^\tau g(x,\varepsilon,w)\partial_x\partial_{\theta_i}\mathbf{m}_\varepsilon\big\rangle,\\
|\tau_3| \leq &\ K\varepsilon\|w\|_{L^\infty}\|\partial_x w\|_{L^2}\|\partial_x\partial_{\theta_i}\mathbf{m}_\varepsilon\|_{L^2}\\
\leq &\ K\|w\|_{L^\infty}\|\partial_x w\|_{L^2}\|\sin\varphi_\varepsilon^\sigma\partial_{\theta_i}\mathbf{m}_\varepsilon\|_{L^2} \text{ by (2.7)}\\
\leq &\ \frac{K}{\sqrt{\varepsilon}}\|w\|_{L^\infty}\|\partial_x w\|_{L^2}.
\end{aligned}$$

Concerning T_{62}, T_{63} and T_{64}, Estimate (4.3) yields:

$$\begin{aligned}
|\langle T_{62}|\partial_{\theta_i}\mathbf{m}_\varepsilon\rangle| \leq & \quad K\varepsilon\|\partial_x w\|_{L^2}^2\|w\|_{L^\infty}\|\partial_{\theta_i}\mathbf{m}_\varepsilon\|_{L^\infty} \\
\leq & \quad K\|\partial_x w\|_{L^2}^2\|w\|_{L^\infty}, \\
|\langle T_{63}|\partial_{\theta_i}\mathbf{m}_\varepsilon\rangle| \leq & \quad K\|\partial_x w\|_{L^2}\|w\|_{L^\infty}\|\partial_{\theta_i}\mathbf{m}_\varepsilon\|_{L^2} \\
\leq & \quad \frac{K}{\sqrt{\varepsilon}}\|\partial_x w\|_{L^2}\|w\|_{L^\infty}, \\
|\langle T_{64}|\partial_{\theta_i}\mathbf{m}_\varepsilon\rangle| \leq & \quad \frac{K}{\varepsilon}\|w\|_{L^\infty}^2\|\partial_{\theta_i}\mathbf{m}_\varepsilon\|_{L^1} \\
\leq & \quad \frac{K}{\varepsilon}\|w\|_{L^\infty}^2.
\end{aligned}$$

Therefore,

$$\left|\varepsilon\langle T_6|\partial_{\theta_i}\mathbf{m}_\varepsilon\rangle\right| \leq K\left(\|w\|_{L^\infty}^2 + \varepsilon\|\partial_x w\|_{L^2}^2\right).$$

So since the norm $\|\cdot\|_\varepsilon$ controls the L^∞ norm, we obtain that

$$|\varepsilon\langle T_6|\partial_{\theta_i}\mathbf{m}_\varepsilon\rangle| \leq K\|w\|_\varepsilon^2.$$

With the same arguments we estimate $\varepsilon\langle T_6|\partial_{\sigma_i}\mathbf{m}_\varepsilon\rangle$ and we conclude the proof of Proposition 4.4.

□

4.2 Equation for w

Plugging (4.7) in (4.1), since $-\frac{h}{\varepsilon}\mathbf{m}_\varepsilon \times e_1 = \sum_{i=1}^{N} h_i\partial_{\theta_i}\mathbf{m}_\varepsilon + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$ and $-\frac{h}{\varepsilon}\mathbf{m}_\varepsilon \times (\mathbf{m}_\varepsilon \times e_1) = \sum_{i=1}^{N}(-1)^i h_i\partial_{\sigma_i}\mathbf{m}_\varepsilon + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$, we obtain the following equation for w:

$$\frac{\partial w}{\partial t} + \nu'(w)\left(\frac{\partial w}{\partial t}\right)\mathbf{m}_\varepsilon = \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}) + \Lambda_\varepsilon + P_\varepsilon w + l^\varepsilon w + Z^\varepsilon(w,\theta,\sigma), \tag{4.8}$$

where

- the linear operator Λ_ε is given by

$$\Lambda_\varepsilon w = -\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w) - \mathbf{m}_\varepsilon \times (\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)),$$

- the linear perturbation due to the applied magnetic field writes

$$P_\varepsilon w = -\frac{h}{\varepsilon}w \times e_1 - \frac{h}{\varepsilon}w_1\mathbf{m}_\varepsilon - \frac{h}{\varepsilon}\sin\varphi_\varepsilon^\sigma w + h\varepsilon\Pi_\varepsilon^{\theta_i}(w)\partial_{\theta_i}\mathbf{m}_\varepsilon + h\varepsilon\Pi_\varepsilon^{\sigma_i}(w)\partial_{\sigma_i}\mathbf{m}_\varepsilon,$$

- l_ε is an exponentially small linear operator on w:

$$\|l^\varepsilon w\|_{L^2} \leq \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})\|w\|_{L^\infty},$$

- the non linear term $H^\varepsilon(w,\theta,\sigma)$ is given by

$$Z^\varepsilon(w,\theta,\sigma) = \quad T_6 - \nu(w)\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}) - \sum_{i=1}^{N} \left(G_\varepsilon^{\theta_i}(\theta_i,\sigma_i,w)\partial_{\theta_i}\mathbf{m}_\varepsilon + G_\varepsilon^{\sigma_i}(\theta_i,\sigma_i,w)\partial_{\sigma_i}\mathbf{m}_\varepsilon \right)$$

$$-\nu(w)\sum_{i=1}^{N}\Big(h_i - h_i\Pi_\varepsilon^{\sigma_i}(w) + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})(w) + G_\varepsilon^{\theta_i}(\theta_i,\sigma_i,w)\Big)\partial_{\theta_i}\mathbf{m}_\varepsilon$$

$$-\nu(w)\sum_{i=1}^{N}\Big((-1)^i h_i - h_i\Pi_\varepsilon^{\sigma_i}(w) + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})(w) + G_\varepsilon^{\sigma_i}(\theta_i,\sigma_i,w)\Big)\partial_{\sigma_i}\mathbf{m}_\varepsilon.$$

In order to inverse the left hand side term in (4.8), we claim without proof (left to the reader) the following lemma:

Lemma 4.1. *There exists ν_0 such that for all $\varepsilon > 0$, for all t, x, if $\|w\|_{L^\infty} \leq \nu_0$ then the linear operator*

$$\begin{array}{rcl} \chi : I\!R^3 & \to & I\!R^3 \\ \xi & \mapsto & \xi + \nu'(w)(\xi)\mathbf{m}_\varepsilon \end{array}$$

is invertible and $\chi^{-1} = Id + V^\varepsilon(w)$ where

$$\|V^\varepsilon(w)\|_{L^\infty} \leq K\|w\|_{L^\infty},$$

where K does not depend neither on ε nor on w.

Using Lemma 4.1 in (4.8) we obtain the following equation for w:

$$\frac{\partial w}{\partial t} = a_\varepsilon + \Lambda_\varepsilon w + P_\varepsilon w + l^\varepsilon w + G^\varepsilon(w,\theta,\sigma) \tag{4.9}$$

where

- $a_\varepsilon = \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$,
- the linear operators Λ_ε and P_ε are defined above,
- l_ε is an exponentially small linear operator on w:

$$\|l^\varepsilon w\|_{L^2} \leq \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})\|w\|_{L^\infty},$$

- the non linear term $G^\varepsilon(w,\theta,\sigma)$ is obtained from $Z_\varepsilon(w,\theta,\sigma)$:

$$G^\varepsilon(w,\theta,\sigma) = Z_\varepsilon(w,\theta,\sigma) + V^\varepsilon(w)\left(\Lambda_\varepsilon w + P_\varepsilon w + l^\varepsilon w + Z_\varepsilon(w,\theta,\sigma)\right).$$

From the expressions of G^ε and Z_ε, with the estimates on T_6, since the $\|\cdot\|_\varepsilon$ norm controls the L^∞ norm, we obtain the following proposition:

Proposition 4.5. *There exists K such that for all ε, for all $\theta \in I\!R^N$, $\sigma \in \Sigma_\delta$ and $w \in \mathcal{W}^\varepsilon_{\theta,\sigma}$, the following estimate holds*

$$\|G^\varepsilon(w,\theta,\sigma)\|_{L^2} \leq K\|w\|_\varepsilon \left(\|\varepsilon\partial_{xx}w\|_{L^2} + \frac{1}{\varepsilon}\|w\|_{L^2} \right). \tag{4.10}$$

5 Meta Stability for a general applied field

In this section, we assume that the applied magnetic field satisfies (1.11) and (1.13) but is not constant in the walls. Starting from Equation (4.1), we aim to obtain an equivalent equation on the new variables (θ,σ,w) by projection on $\partial_{\theta_i}\mathbf{m}_\varepsilon$ and $\partial_{\sigma_i}\mathbf{m}_\varepsilon$. We proceed as in Sections 3 and 4, and we only detail the changes due to the fact that Assumption (1.14) is not yet satisfied.

5.1 New coordinates for the non constant applied field case

The applied field h appears in the terms T_2 and T_4 of equation (4.1).
We denote by $h_i(t) = h(t, \sigma_i(t))$, and we write the Taylor expansion of h in the wall i,

$$h(t,x) = h_i(t) + (x-\sigma_i)\gamma_i(t) + (x-\sigma_i)^2 K_i(t,x),$$

where, by assumption (1.11),

$$|K_i(t,x)| \leq M \text{ in a neighborhood of } \sigma_i. \tag{5.11}$$

Concerning T_2, on the one hand, we have:

$$\big\langle T_2 | \partial_{\theta_i}\mathbf{m}_\varepsilon \big\rangle = \big\langle h\partial_{\theta_i}\mathbf{m}_\varepsilon | \partial_{\theta_i}\mathbf{m}_\varepsilon \big\rangle + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}).$$

Now, using the Taylor expansion of h, we have that

$$\begin{aligned}\big\langle h\partial_{\theta_i}\mathbf{m}_\varepsilon | \partial_{\theta_i}\mathbf{m}_\varepsilon \big\rangle = \quad & h_i(t)\big\langle \partial_{\theta_i}\mathbf{m}_\varepsilon | \partial_{\theta_i}\mathbf{m}_\varepsilon \big\rangle + \gamma_i(t)\int_{\sigma_i-\delta}^{\sigma_i+\delta}(x-\sigma_i)|\partial_{\theta_i}\mathbf{m}_\varepsilon|^2 dx \\ & + \int_{\sigma_i-\delta}^{\sigma_i+\delta}(x-\sigma_i)^2 K_i(t,x)|\partial_{\theta_i}\mathbf{m}_\varepsilon|^2.\end{aligned}$$

The second term in the right hand side is vanishing since $\partial_{\theta_i}\mathbf{m}_\varepsilon$ is even, so we obtain that

$$\big\langle T_2 | \partial_{\theta_i}\mathbf{m}_\varepsilon \big\rangle = \frac{2}{\varepsilon}h_i + 2\varepsilon r_\varepsilon^{\theta_i}(t),$$

with

$$r_\varepsilon^{\theta_i}(t) = \frac{1}{2\varepsilon}\int_{\sigma_i-\delta}^{\sigma_i+\delta}(x-\sigma_i)^2 K_i(t,x)|\partial_{\theta_i}\mathbf{m}_\varepsilon|^2 + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}).$$

This term is bounded, that is there exists K such that $|r_\varepsilon^{\theta_i}(t)| \leq K$. Indeed, using (5.11) and the estimates on $\partial_{\theta_i}\mathbf{m}_\varepsilon$ in Section 2.1:

$$\begin{aligned}\left|\frac{1}{2\varepsilon}\int_{\sigma_i-\delta}^{\sigma_i+\delta}(x-\sigma_i)^2 K_i(t,x)|\partial_{\theta_i}\mathbf{m}_\varepsilon|^2\right| \leq \quad & \frac{M}{2\varepsilon}\int_{-\delta}^{\delta}\frac{1}{\varepsilon^2}\frac{|x|^2}{\cosh^2(x/\varepsilon)}dx \\ \leq \quad & M\int_{-\infty}^{\infty}\frac{z^2}{\cosh^2 z}dz.\end{aligned}$$

On the other hand, by the same arguments, we have:

$$\big\langle T_2 | \partial_{\sigma_i}\mathbf{m}_\varepsilon \big\rangle = (-1)^i\frac{2}{\varepsilon}h_i + 2\varepsilon r_\varepsilon^{\sigma_i}(t),$$

with $|r_\varepsilon^{\sigma_i}(t)| \leq K$.

Concerning T_4, we introduce the operators $\Pi_\varepsilon^{h,\theta_i}$ and $\Pi_\varepsilon^{h,\sigma_i}$ defined for $w \in \mathcal{W}_{\theta,\sigma}^\varepsilon$ by

$$\Pi_\varepsilon^{h,\theta_i}(w) = \big\langle h\sin\varphi_\varepsilon^\sigma w | \partial_{\theta_i}\mathbf{m}_\varepsilon \big\rangle \quad \text{and } \Pi_\varepsilon^{h,\sigma_i}(w) = \big\langle h\sin\varphi_\varepsilon^\sigma w | \partial_{\sigma_i}\mathbf{m}_\varepsilon \big\rangle.$$

We have

$$\big\langle T_4 | \partial_{\theta_i}\mathbf{m}_\varepsilon \big\rangle = \frac{(-1)^{i+1}}{\varepsilon}\Pi_\varepsilon^{h,\sigma_i}(w) - \frac{1}{\varepsilon}\Pi_\varepsilon^{h,\theta_i}(w) + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})(w),$$

and

$$\big\langle T_4 | \partial_{\sigma_i}\mathbf{m}_\varepsilon \big\rangle = (-1)^i\frac{1}{\varepsilon}\Pi_\varepsilon^{h,\theta_i}(w) + \frac{1}{\varepsilon}\Pi_\varepsilon^{h,\sigma_i}(w) + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})(w).$$

As in Proposition 4.2, using assumption (1.13), we prove that there exists a constant K such that

$$\left|\Pi_\varepsilon^{h,\theta_i}(w)\right| + \left|\Pi_\varepsilon^{h,\sigma_i}(w)\right| \le K\|w\|_{L^\infty}. \tag{5.12}$$

Using the same notations as in (4.6) we obtain that

$$\left(I + L^i(w) + Q^i(w) + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})\right)\frac{dv_i}{dt} = h_i \begin{pmatrix} 1 \\ (-1)^i \end{pmatrix} + \varepsilon^2 r_\varepsilon^i + \lambda^i(w) + F^i, \tag{5.13}$$

where

$$r_\varepsilon^i = \begin{pmatrix} r_\varepsilon^{\theta_i} \\ r_\varepsilon^{\sigma_i} \end{pmatrix} \text{ and } \lambda^i(w) = \begin{pmatrix} -\frac{1}{2}\Pi_\varepsilon^{h,\theta_i}(w) + \frac{(-1)^{i+1}}{2}\Pi_\varepsilon^{h,\sigma_i}(w) \\ -\frac{1}{2}\Pi_\varepsilon^{h,\sigma_i}(w) + \frac{(-1)^i}{2}\Pi_\varepsilon^{h,\theta_i}(w) \end{pmatrix} + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})(w).$$

With Proposition 4.3 we obtain that $v_i = (\theta_i, \sigma_i)$ satisfies

$$\frac{dv_i}{dt} = h_i \begin{pmatrix} 1 \\ (-1)^i \end{pmatrix} + \varepsilon^2 r_\varepsilon^i + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}) + \mathcal{L}^i(w) + \mathcal{G}^i(w), \tag{5.14}$$

where

$$\mathcal{L}^i(w) = \lambda^i(w) - h_i L^i(w)\begin{pmatrix} 1 \\ (-1)^i \end{pmatrix} - \varepsilon^2 L^i(w) r_\varepsilon^i + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})(w) := \mathcal{L}_1^i(w) + \mathcal{L}_2^i(w) + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})(w)$$

with

$$\mathcal{L}_1^i(w) = \begin{pmatrix} -\frac{1}{2}\left(\Pi_\varepsilon^{h,\theta_i}(w) + h_i\Pi_\varepsilon^{\theta_i}(w)\right) + \frac{(-1)^{i+1}}{2}\left(\Pi_\varepsilon^{h,\sigma_i}(w) - h_i\Pi_\varepsilon^{\sigma_i}(w)\right) \\ \\ \frac{(-1)^i}{2}\left(\Pi_\varepsilon^{h,\theta_i}(w) - h_i\Pi_\varepsilon^{\theta_i}(w)\right) - \frac{1}{2}\left(\Pi_\varepsilon^{h,\sigma_i}(w) + h_i\Pi_\varepsilon^{\sigma_i}(w)\right) \end{pmatrix},$$

$$\mathcal{L}_2^i(w) = -\frac{(-1)^i\varepsilon^2}{4}\begin{pmatrix} r_\varepsilon^{\theta_i}\Pi_\varepsilon^{\sigma_i}(w) + r_\varepsilon^{\sigma_i}\Pi_\varepsilon^{\theta_i}(w) \\ \\ r_\varepsilon^{\theta_i}\Pi_\varepsilon^{\theta_i}(w) + r_\varepsilon^{\sigma_i}\Pi_\varepsilon^{\sigma_i}(w) \end{pmatrix},$$

and where

$$\mathcal{G}^i(w) = (I + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}))F^i + \tilde{Q}^i(w)\left(h_i\begin{pmatrix} 1 \\ (-1)^i \end{pmatrix} + \varepsilon^2 r_\varepsilon^i + \lambda^i(w) + F^i\right) - L^i(w)\left(\lambda^i(w) + F^i\right).$$

The linear part $\mathcal{L}^i(w)$ and the non linear term $\mathcal{G}^i$ can be estimated as the linear and the non linear parts of (4.7) (see Propositions 4.2 and 4.4):

Proposition 5.1. *There exists a constant K independent of $\varepsilon > 0$ such that for all $\theta \in I\!R^N$ and $\sigma \in \Sigma_\delta$, for all $w \in \mathcal{W}_{\theta,\sigma}^\varepsilon$,*

$$|\mathcal{L}^i(w)| \le K\|w\|_\varepsilon \text{ and } |\mathcal{G}^i(w)| \le K\|w\|_\varepsilon^2.$$

Now when we plug (5.14) in (4.1), the term T_2 is not yet cancelled by $h_i\partial_{\theta_i}\mathbf{m}_\varepsilon + (-1)^i h_i\partial_{\sigma_i}\mathbf{m}_\varepsilon$, and we obtain:

$$\begin{aligned}\frac{\partial w}{\partial t} + \nu'(w)(\frac{\partial w}{\partial t})\mathbf{m}_\varepsilon = \quad & \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}) + \sum_{i=1}^N \left(h - h_i + \varepsilon^2 r_\varepsilon^{\theta_i}\right)\partial_{\theta_i}\mathbf{m}_\varepsilon \\ & + \sum_{i=1}^N \left(h - h_i + \varepsilon^2 r_\varepsilon^{\sigma_i}\right)\partial_{\sigma_i}\mathbf{m}_\varepsilon + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})(w) \\ & + \Lambda^\varepsilon w + \overline{P^\varepsilon}(w) + \overline{Z^\varepsilon}(w,\theta,\sigma),\end{aligned}$$

with

$$\overline{P^\varepsilon}(w) = -\frac{h}{\varepsilon}w\times e_1 - \frac{h}{\varepsilon}(\sin\varphi_\varepsilon^\sigma w + w_1\mathbf{m}_\varepsilon) - \mathcal{L}_1^i(w)\partial_{\theta_i}\mathbf{m}_\varepsilon - \mathcal{L}_2^i(w)\partial_{\sigma_i}\mathbf{m}_\varepsilon,$$

and

$$\begin{aligned}\overline{Z^\varepsilon}(w,\theta,\sigma) = \quad & T_6 - \nu(w)\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}) - \sum_{i=1}^N \left(\mathcal{G}^{i,1}(w)\partial_{\theta_i}\mathbf{m}_\varepsilon + \mathcal{G}i,2(w)\partial_{\sigma_i}\mathbf{m}_\varepsilon\right)\\ & -\nu(w)\sum_{i=1}^N \left(h_i + \varepsilon^2 r_\varepsilon^{\theta_i} + \mathcal{L}^{i,1}(w) + \mathcal{G}^{i,1}(w)\right)\partial_{\theta_i}\mathbf{m}_\varepsilon\\ & -\nu(w)\sum_{i=1}^N \left((-1)^i h_i + \varepsilon^2 r_\varepsilon^{\sigma_i} + \mathcal{L}^{i,2}(w) + \mathcal{G}^{i,2}(w)\right)\partial_{\sigma_i}\mathbf{m}_\varepsilon.\end{aligned}$$

By Using Lemma 4.3, we obtain the following equation on w:

$$\frac{\partial w}{\partial t} = A_\varepsilon + \Lambda^\varepsilon w + \overline{P^\varepsilon}(w) + l^\varepsilon(w) + \overline{G^\varepsilon}(w,\theta,\sigma) \tag{5.15}$$

where

$$A_\varepsilon = \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}) + \sum_{i=1}^N \left(h - h_i + \varepsilon^2 r_\varepsilon^{\theta_i}\right)\partial_{\theta_i}\mathbf{m}_\varepsilon + \sum_{i=1}^N \left(h - h_i + \varepsilon^2 r_\varepsilon^{\sigma_i}\right)\partial_{\sigma_i}\mathbf{m}_\varepsilon,$$

$$l^\varepsilon(w) = \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})(w) + V^\varepsilon(w)\left(\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}) + \sum_{i=1}^N \left(h - h_i + \varepsilon^2 r_\varepsilon^{\theta_i}\right)\partial_{\theta_i}\mathbf{m}_\varepsilon + \sum_{i=1}^N \left(h - h_i + \varepsilon^2 r_\varepsilon^{\sigma_i}\right)\partial_{\sigma_i}\mathbf{m}_\varepsilon\right),$$

and

$$\overline{G^\varepsilon}(w,\theta,\sigma) = \overline{Z^\varepsilon}(w,\theta,\sigma) + V(w)\left(\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})(w) + \Lambda^\varepsilon w + \overline{P^\varepsilon}(w) + \overline{Z^\varepsilon}(w,\theta,\sigma)\right).$$

We have the following estimates:

Proposition 5.2. *There exists a constant K such that for all $\theta\in I\!R^N$, for all $\sigma\in\Sigma_\delta$, for all $w\in\mathcal{W}_{\theta,\sigma}^\varepsilon$,*

- $\|A_\varepsilon\|_{L^2}\leq K\sqrt{\varepsilon}$,
- $\|l^\varepsilon(w)\|_{L^2}\leq K\sqrt{\varepsilon}\|w\|_\varepsilon$,
- $\|\overline{G^\varepsilon}(w,\theta,\sigma)\|_{L^2}\leq K\|w\|_\varepsilon\left(\|\varepsilon\partial_{xx}w\|_{L^2} + \frac{1}{\varepsilon}\|w\|_{L^2}\right)$.

Proof. We remark that, using the Taylor expansion of h in the wall i, there exists a constant K such that $|h(t,x)-h_i(t)|\leq K|x-\sigma_i|$. So

$$\begin{aligned}\|(h-h_i)\partial_{\theta_i}\mathbf{m}_\varepsilon\|_{L^2}^2 \leq \quad & K\int_{\sigma_i-\delta}^{\sigma_i+\delta}(x-\sigma_i)^2|\partial_{\theta_i}\mathbf{m}_\varepsilon|^2\\ \leq \quad & K\int_{\sigma_i-\delta/2}^{\sigma_i+\delta/2}(x-\sigma_i)^2\frac{1}{\varepsilon^2\cosh^2(\frac{x-\sigma_i}{\varepsilon})}dx + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})\\ \leq \quad & K\varepsilon\int_{I\!R}\frac{z^2}{\cosh^2 z}dz + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})\\ \leq \quad & K\varepsilon.\end{aligned}$$

In addition,

$$\|\varepsilon^2 r_\varepsilon^{\theta_i}\partial_{\theta_i}\mathbf{m}_\varepsilon\|_{L^2}\leq\varepsilon^2\|\partial_{\theta_i}\mathbf{m}_\varepsilon\|_{L^2}\leq K\varepsilon^{\frac{3}{2}}$$

with Proposition 2.2. The same estimates hold for the term concerning $\partial_{\sigma_i}\mathbf{m}_\varepsilon$ in A_ε. Therefore

$$\|A_\varepsilon\|_{L^2} \leq K\sqrt{\varepsilon}.$$

Concerning $l^\varepsilon(w)$, we have:

$$\begin{aligned}
\|l^\varepsilon(w)\|_{L^2} \leq \quad & \|V^\varepsilon(w)\|_{L^\infty}\Bigg(\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}) + \sum_{i=1}^N \|(h - h_i + \varepsilon^2 r_\varepsilon^{\theta_i})\partial_{\theta_i}\mathbf{m}_\varepsilon\|_{L^2} \\
& + \sum_{i=1}^N \|(h - h_i + \varepsilon^2 r_\varepsilon^{\sigma_i})\partial_{\sigma_i}\mathbf{m}_\varepsilon\|_{L^2}\Bigg) \\
\leq \quad & C\sqrt{\varepsilon}\|w\|_{L^\infty} \text{ from Lemma 4.1} \\
\leq \quad & C\sqrt{\varepsilon}\|w\|_\varepsilon.
\end{aligned}$$

The expression of $\overline{G^\varepsilon}(w,\theta,\sigma)$, the estimates on T_6 together with Corollary 3.2 yield the claimed estimate on $\overline{G^\varepsilon}$.

□

5.2 Estimates for Theorem 1.1

We take the inner product of (5.15) with $\mathcal{H}^\varepsilon(w)$. Since

$$\frac{1}{2}\frac{d}{dt}\big\langle w|\mathcal{H}^\varepsilon(w)\big\rangle = \big\langle \frac{dw}{dt}|\mathcal{H}^\varepsilon(w)\big\rangle + \frac{1}{2}\sum_{i=1}^N \frac{d\sigma_i}{dt}\big\langle \partial_{\sigma_i} f_\varepsilon^\sigma w|w\big\rangle,$$

we obtain:

$$\frac{1}{2}\frac{d}{dt}\big\langle w|\mathcal{H}^\varepsilon(w)\big\rangle + \|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}^2 = \bar{M}_1 + \ldots + \bar{M}_5, \tag{5.16}$$

with

- $\bar{M}_1 = \big\langle A_\varepsilon|\mathcal{H}^\varepsilon(w)\big\rangle$,
- $\bar{M}_2 = \overline{P^\varepsilon}(w)|\mathcal{H}^\varepsilon(w)\big\rangle + \frac{1}{2}\sum_{i=1}^N (-1)^i h_i \big\langle \partial_{\sigma_i} f_\varepsilon^\sigma w|w\big\rangle$,
- $\bar{M}_3 = \frac{1}{2}\sum_{i=1}^N (\varepsilon^2 r_\varepsilon^{\sigma_i} + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}))\big\langle \partial_{\sigma_i} f_\varepsilon^\sigma w|w\big\rangle$,
- $\bar{M}_4 = \frac{1}{2}\sum_{i=1}^N (\mathcal{L}_2^i(w) + \mathcal{G}_2^i(w))\big\langle \partial_{\sigma_i} f_\varepsilon^\sigma w|w\big\rangle$,
- $\bar{M}_5 = \big\langle \overline{G^\varepsilon}(w,\theta,\sigma)|\mathcal{H}^\varepsilon(w)\big\rangle$.

Using Proposition 5.2 we have

$$|\bar{M}_1| \leq K\sqrt{\varepsilon}\|m^\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2},$$

and

$$|\bar{M}_5| \leq K\|w\|_\varepsilon \|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}^2.$$

We have

$$\begin{aligned}|\bar{M}_3| \leq \quad & K\varepsilon^2 \sum_{i=1}^{N} \|\partial_{\sigma_i} f_\varepsilon^\sigma\|_{L^\infty} \|w\|_{L^2}^2 \\ \leq \quad & K\|w\|_{L^2}^2 \text{ since } \|\partial_{\sigma_i} f_\varepsilon^\sigma\|_{L^\infty} \leq \frac{K}{\varepsilon^2} \\ \leq \quad & K\varepsilon^2 \|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}^2 \text{ with Corollary 3.2.}\end{aligned}$$

In addition,

$$\begin{aligned}|\bar{M}_4| \leq \quad & \tfrac{1}{2}\textstyle\sum_{i=1}^{N} |\mathcal{L}_2^i(w) + \mathcal{G}_2^i(w)| \|\partial_{\sigma_i} f_\varepsilon^\sigma\|_{L^\infty} \|w\|_{L^2}^2 \\ \leq \quad & K\|w\|_\varepsilon \frac{1}{\varepsilon^2} \|w\|_{L^2}^2 \text{ by Proposition 5.1} \\ \leq \quad & K\|w\|_\varepsilon \|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}^2 \text{ with Corollary 3.2.}\end{aligned}$$

In order to estimate $\bar{M}_2$ we aim to split $\bar{M}_2$: $\bar{M}_2 = \bar{M}_{21} + \bar{M}_{22}$ with

$$\bar{M}_{21} = \big\langle -\frac{h}{\varepsilon} w \times e_1 | \mathcal{H}^\varepsilon(w) \big\rangle - \big\langle \frac{h}{\varepsilon} w_1 \mathbf{m}_\varepsilon | \mathcal{H}^\varepsilon(w) \big\rangle,$$

$$\bar{M}_{22} = -\big\langle \frac{h}{\varepsilon} \sin\varphi_\varepsilon^\sigma w | \mathcal{H}^\varepsilon(w) \big\rangle + \frac{1}{2} \sum_{i=1}^{N} h_i \big\langle \partial_{\sigma_i} f_\varepsilon^\sigma w | w \big\rangle.$$

The first term is estimated as in Section 3.3 (see terms P_1 and P_2):

$$|\bar{M}_{21}| \leq K\sqrt{\varepsilon} \|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}^2.$$

To estimate $\bar{M}_{22}$, we use the IMS formula. In the wall i, we split h in $(h - h_i) + h_i$. The part $\big\langle \frac{h_i}{\varepsilon} \sin\varphi_\varepsilon^\sigma \chi_i w | \mathcal{H}^\varepsilon(\chi_i w) \big\rangle$ will be estimated together with $h_i \big\langle \partial_{\sigma_i} f_\varepsilon^\sigma \chi_i w | \chi_i w \big\rangle$ as in Section 3.3. In order to argue that the remainder term with $h - h_i$ is small, we must reduce the support of χ_i. Therefore we use another system of cut-off functions.

Let us introduce $\bar{\chi}_0, \ldots, \bar{\chi}_N$ such that

- supp $\bar{\chi}_0 \subset [0, L] \setminus \bigcup_{i=1}^{N} [\sigma_i - \sqrt{\varepsilon}, \sigma_i + \sqrt{\varepsilon}]$,
- supp $\bar{\chi}_i \subset [\sigma_i - 2\sqrt{\varepsilon}, \sigma_i + 2\sqrt{\varepsilon}]$ for $i \neq 0$,
- $\sum_{i=0}^{N} \bar{\chi}_i^2 = 1$,
- $\|\partial_x \bar{\chi}_i\|_{L^\infty} \leq \frac{C}{\sqrt{\varepsilon}}$,
- $\|\partial_{xx} \bar{\chi}_i\|_{L^\infty} \leq \frac{C}{\varepsilon}$.

We plug the cut-off functions in $\bar{M}_{22}$. We obtain:

$$\begin{aligned}\bar{M}_{22} = \quad & -\sum_{i=0}^{N} \big\langle \frac{h}{\varepsilon} \sin\varphi_\varepsilon^\sigma w \bar{\chi}_i^2 | \mathcal{H}^\varepsilon(w) \big\rangle + \frac{1}{2} \sum_{i=1}^{N} h_i \big\langle \partial_{\sigma_i} f_\varepsilon^\sigma w \bar{\chi}_i^2 | w \big\rangle \\ & + \frac{1}{2} \sum_{i=1}^{N} h_i \big\langle \partial_{\sigma_i} f_\varepsilon^\sigma w \bar{\chi}_0^2 | w \big\rangle,\end{aligned}$$

since now the support of $\partial_{\sigma_i} f_\varepsilon^\sigma$ intersects the support of $\bar\chi_0$.
As we remarked in Section 3.3,

$$-\sum_{i=0}^{N}\left\langle\frac{h}{\varepsilon}\sin\varphi_\varepsilon^\sigma w\bar\chi_i^2|\mathcal{H}^\varepsilon(w)\right\rangle = -\sum_{i=0}^{N}\left\langle\frac{h}{\varepsilon}\sin\varphi_\varepsilon^\sigma\bar\chi_i|\mathcal{H}^\varepsilon(\bar\chi_i w)\right\rangle$$
$$-\left\langle h\sum_{i=0}^{N}|\partial_x\bar\chi_i|^2\sin\varphi_\varepsilon^\sigma w|w\right\rangle.$$

Hence we split $\bar M_{22}$ is the following way:

$$\bar M_{22} = \Psi_0+\Psi_1+\Psi_2+\sum_{i=1}^{N}\Psi_{3i}+\Psi_4,$$

with

- $\Psi_0 = \left\langle -\frac{h}{\varepsilon}\sin\varphi_\varepsilon^\sigma\bar\chi_0 w|\mathcal{H}^\varepsilon(\bar\chi_0 w)\right\rangle$,
- $\Psi_1 = -\left\langle h\sum_{i=0}^{N}|\partial_x\bar\chi_i|^2\sin\varphi_\varepsilon^\sigma w|w\right\rangle$,
- $\Psi_2 = -\sum_{i=1}^{N}\left\langle\frac{h-h_i}{\varepsilon}\sin\varphi_\varepsilon^\sigma\bar\chi_i w|\mathcal{H}^\varepsilon(\bar\chi_i w)\right\rangle$,
- $\Psi_{3i} = \sum_{i=1}^{N}h_i\left(-\left\langle\frac{\sin\varphi_\varepsilon^\sigma}{\varepsilon}\bar\chi_i w|\mathcal{H}^\varepsilon(\bar\chi_i w)\right\rangle+\frac{(-1)^i}{2}\left\langle\partial_{\sigma_i}f_\varepsilon^\sigma\bar\chi_i w|\bar\chi_i w\right\rangle\right)$,
- $\Psi_4 = \frac{1}{2}\sum_{i=1}^{N}h_i\left\langle\partial_{\sigma_i}f_\varepsilon^\sigma w\bar\chi_0^2|w\right\rangle$.

Concerning the term Ψ_0, we remark that on the support of $\bar\chi_0$,

$$\mathbf{m}_\varepsilon = \pm e_1+\mathcal{O}(e^{-\frac{c}{\sqrt{\varepsilon}}}),$$

$$f_\varepsilon^\sigma = \frac{1}{\varepsilon}+\mathcal{O}(e^{-\frac{c}{\sqrt{\varepsilon}}}).$$

Therefore with the same arguments as for I_0 in Section 3.3, we obtain that:

$$|\Psi_0| \le (1+\mathcal{O}(e^{-\frac{c}{\sqrt{\varepsilon}}}))\|h\|_{L^\infty}\|\mathbf{m}_\varepsilon\times\mathcal{H}^\varepsilon(\bar\chi_0 w)\|_{L^2}^2. \tag{5.17}$$

For the term Ψ_1, we remark that $\|\bar\chi_i\|_{L^\infty}\le\frac{C}{\sqrt{\varepsilon}}$ so that

$$|\Psi_1| \le C\frac{\|h\|_{L^\infty}}{\varepsilon}\|w\|_{L^2}^2 \le C\varepsilon\|\mathbf{m}_\varepsilon\times\mathcal{H}^\varepsilon(w)\|_{L^2}^2, \tag{5.18}$$

with Corollary 3.2.
In the support of $\bar\chi_i$ then, with Assumption (1.13), $|h-h_i|\le K\sqrt{\varepsilon}$, so

$$|\Psi_2| \le \frac{c}{\sqrt{\varepsilon}}\|\bar\chi_i w\|_{L^2}\|\mathbf{m}_\varepsilon\times\mathcal{H}^\varepsilon(\bar\chi_i w)\|_{L^2} \le C\sqrt{\varepsilon}\|\mathbf{m}_\varepsilon\times\mathcal{H}^\varepsilon(\bar\chi_i w)\|_{L^2}^2, \tag{5.19}$$

The terms Ψ_{3i} are estimated as the terms $I_{1,i}$ in Section 3.3:

$$|\Psi_{3i}| \le \|h\|_{L^\infty}\|\mathbf{m}_\varepsilon\times\mathcal{H}^\varepsilon(\bar\chi_i w)\|_{L^2}^2. \tag{5.20}$$

The last term Ψ_4 is small since on the support of $\bar{\chi}_0$, $\partial_x f_\varepsilon^\sigma = \mathcal{O}(e^{-\frac{c}{\sqrt{\varepsilon}}})$, so

$$|\Psi_4| \leq \mathcal{O}(e^{-\frac{c}{\sqrt{\varepsilon}}})\|w\|_{L^2}^2 \leq \mathcal{O}(e^{-\frac{c}{\sqrt{\varepsilon}}})\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}^2. \tag{5.21}$$

Combining Estimates (5.18), (5.19), (5.20) and (5.21) we obtain that

$$\left|\bar{M}_{22}\right| \leq (1 + \mathcal{O}(e^{-\frac{c}{\sqrt{\varepsilon}}})\|h\|_{L^\infty} \sum_{i=0}^{N} \|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(\bar{\chi}_i w)\|_{L^2}^2 + C\sqrt{\varepsilon}\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}^2. \tag{5.22}$$

It remains to compare $\sum_{i=0}^{N} \|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(\bar{\chi}_i w)\|_{L^2}^2$ with $\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}^2$. As in the end of Section 3.3 we have:

$$\mathcal{H}^\varepsilon(\bar{\chi}_i w) = \bar{\chi}_i \mathcal{H}^\varepsilon(w) + \bar{\tau}_i$$

with $\bar{\tau}_i = -2\varepsilon\partial_x\bar{\chi}_i\partial_x w - \varepsilon\partial_{xx}\bar{\chi}_i w$, and

$$\begin{aligned}
\|\bar{\tau}_i\|_{L^2} \leq\ & c\varepsilon\|\partial_x\bar{\chi}_i\|_{L^\infty}\|\partial_x w\|_{L^2} + c\varepsilon\|\partial_{xx}\bar{\chi}_i\|_{L^\infty}\|w\|_{L^2} \\
\leq\ & c\sqrt{\varepsilon}\|\partial_x w\|_{L^2} + c\|w\|_{L^2} \text{ from the estimates on the derivatives of } \bar{\chi}_i \\
\leq\ & c\sqrt{\varepsilon}\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2} \text{ with Corollary 3.2.}
\end{aligned}$$

Therefore with the same method as in Section 3.3, we obtain that

$$\sum_{i=0}^{N} \|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(\bar{\chi}_i w)\|_{L^2}^2 \leq (1 + c\sqrt{\varepsilon})\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}^2.$$

Hence we obtain that:

$$\left|\bar{M}_{22}\right| \leq \left(\|h\|_{L^\infty} + c\sqrt{\varepsilon}\right)\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}^2. \tag{5.23}$$

From (5.16), we obtain that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\langle w|\mathcal{H}^\varepsilon(w)\rangle + \|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}^2 \leq\ & C\sqrt{\varepsilon}\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2} \\
& + \left(\|h\|_{L^\infty} + c\sqrt{\varepsilon} + K\|w\|_\varepsilon\right)\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}^2.
\end{aligned}$$

Now we assumed that $\|h\|_{L^\infty} \leq 1 - 3\tau$ so we obtain, using the Young inequality for the first right hand side term:

$$\frac{1}{2}\frac{d}{dt}\langle w|\mathcal{H}^\varepsilon(w)\rangle + \left(2\tau - c\sqrt{\varepsilon} - K\|w\|_\varepsilon\right)\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}^2 \leq \frac{C}{\tau}\varepsilon + \tau\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}^2$$

that is

$$\frac{1}{2}\frac{d}{dt}\langle w|\mathcal{H}^\varepsilon(w)\rangle + \left(\tau - c\sqrt{\varepsilon} - K\|w\|_\varepsilon\right)\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}^2 \leq \frac{C}{\tau}\varepsilon.$$

So for ε small enough, while σ remains in Σ_δ, we have:

$$\frac{1}{2}\frac{d}{dt}\langle w|\mathcal{H}^\varepsilon(w)\rangle + \left(\frac{\tau}{2} - K\|w\|_\varepsilon\right)\|\mathbf{m}_\varepsilon \times \mathcal{H}^\varepsilon(w)\|_{L^2}^2 \leq \frac{C}{\tau}\varepsilon,$$

and while σ remains in Σ_δ, while $\|w\|_\varepsilon \leq \frac{\tau}{4K}$, using Proposition 3.2, we have:

$$\frac{1}{2}\frac{d}{dt}\langle w|\mathcal{H}^\varepsilon(w)\rangle + \frac{\tau(1-\tau)}{4\varepsilon}\langle w|\mathcal{H}^\varepsilon(w)\rangle \leq \frac{C}{\tau}\varepsilon.$$

By comparison argument, we obtain then that while σ remains in Σ_δ, while $\|w\|_\varepsilon \leq \frac{\tau}{4K}$

$$\langle w|\mathcal{H}^\varepsilon(w)\rangle \leq \frac{4C}{\tau(1-\tau)}e^2 + \langle w_0|\mathcal{H}^\varepsilon(w_0)\rangle e^{-\frac{\tau(1-\tau)}{2\varepsilon}t}.$$

Plugging this inequality in (5.14) we obtain that while σ remains in Σ_δ, while $\|w\|_\varepsilon \leq \frac{\tau}{4K}$,

$$\left|\frac{d}{dt}(\theta_i - \theta_i^{ref})\right| + \left|\frac{d}{dt}(\sigma_i - \sigma_i^{ref})\right| \leq K\varepsilon^2 + C\|w_0\|_\varepsilon^2 e^{-\frac{\tau(1-\tau)}{2\varepsilon}t},$$

and by integration in time we obtain

$$\left|\theta_i - \theta_i^{ref}\right|(t) + \left|\sigma_i - \sigma_i^{ref}\right|(t) \leq \left|\theta_i - \theta_i^{ref}\right|(0) + \left|\sigma_i - \sigma_i^{ref}\right|(0) + K\varepsilon^2 t + K\varepsilon\|w_0\|_\varepsilon^2.$$

So on the time interval $[0, \frac{1}{\varepsilon}]$, we have that while σ remains in Σ_δ, while $\|w\|_\varepsilon \leq \frac{\tau}{4K}$,

$$\|w(t)\|_\varepsilon \leq c\|w_0\|_\varepsilon + K\varepsilon$$

and

$$\left|\theta_i - \theta_i^{ref}\right|(t) + \left|\sigma_i - \sigma_i^{ref}\right|(t) \leq \left|\theta_i - \theta_i^{ref}\right|(0) + \left|\sigma_i - \sigma_i^{ref}\right|(0) + K\varepsilon + K\varepsilon\|w_0\|_\varepsilon^2,$$

so we conclude the proof of Theorem 1.2 with the same arguments as for Theorem 1.3.

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