Domain walls dynamics in a nanowire subject to an electric current

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Abstract. In this work, we aim to study a one dimensional model of ferromagnetic wire submitted to an electric field modeled by a transport term involved in the Landau-Lifschitz equation. We will consider two types of wires: the case of a wire with elliptical section and the case of a wire with round section. For both cases we prove the stability of exact solutions describing one wall configurations.

Keywords: Landau-Lifschitz equation, ferromagnetic materials, stability.

1 Introduction

Ferromagnetic materials exhibit a strong attraction to magnetic fields. They are able to retain their magnetic properties after vanishing of the external field. This particularity gives them important properties for applications in many industrial sectors as radar protection, storage of information, energy management and telecommunications equipment (see [3], [9], [10] and [18] for more informations).

One of the most promising applications of ferromagnetic nanowires is the digital data storage in "racetrack memories" (see [16]). The formation of magnetic domains, in which the magnetization is along the wire, either in one sense or in the other sense, allows the storage of digital informations. The domains are separated by domain walls, thin zones in which the magnetization presents large variations. The information is transported along the wire (for example to a reading head) by an electric current inducing walls motion. Compared to an applied magnetic field, this solution can be very useful. Indeed it is easier to generate a constant electric current in a wire, even if it is not straight. Moreover, a constant applied current induces a motion of the walls preserving their positions one with respect to each other, while an application of a constant magnetic field in a finite wire can induce the collapse of consecutive walls and consequently the annihilation of domains.

In this paper we address the description of the effects of an electric current in a ferromagnetic material for a one dimensional model of infinite wire. In particular we will consider one wall configurations in the case of wires with round cross section or with elliptical cross section. For both case, we will prove the stability of such configurations.

Let us describe the one dimensional model we deal with.

A ferromagnetic material is characterized by a spontaneous magnetization represented by a magnetic moment. We consider an infinite homogeneous nanowire assimilated to the real line $\mathbb{R}e_1$, where (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$. We denote by m the magnetization:

$$\begin{array}{rccc} m: & \mathbb{R}_+ \times \mathbb{R} & \to & \mathbb{R}^3 \\ & (t,x) & \mapsto & m(t,x). \end{array}$$

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The magnetic moment m, the magnetic induction B and the magnetic field H are linked by the following constitutive relation:

$$B = H + \overline{m}$$

where B and H are defined on the whole space $\mathbb{R}^3$ and where $\overline{m}$ is the extension of m by zero outside the ferromagnetic domain.

Furthermore we assume that the studied material is saturated, so that the magnetic moment m takes its value in S^2 the unit sphere of $\mathbb{R}^3$. In the case of a ferromagnetic nanowire submitted to an electric current, Thiaville, Miltat, Nakatani and Susuki have proposed in [17] a process to integrate electric current effect on ferromagnetic materials in the Landau-Lifshitz equation, adding a transport term of the form $(v \cdot \nabla)m + m \times ((v \cdot \nabla)m)$ modeling the electric current, where $v(t,x)$ is a vector field directed along the direction of electrons motion, with an amplitude proportional to the current density.

Therefore, in the case of a one dimensional model of nanowire, the behavior of magnetic moment m is described by the following Landau-Lifschitz type equation:

$$\frac{\partial m}{\partial t} = -m \times H_e(m) - m \times (m \times H_e(m)) + v\frac{\partial m}{\partial x} + m \times v\frac{\partial m}{\partial x}, \tag{1}$$

where H_e, the effective field derived from micromagnetism energy (see [1]) is given by:

$$H_e(m) = \frac{\partial^2 m}{\partial x^2} + h_d(m).$$

The term $\dfrac{\partial^2 m}{\partial x^2}$ is called the exchange field and $h_d(m)$ represents the demagnetizing field. In the sequel, we denote $\partial_x = \dfrac{\partial}{\partial x}$ and $\partial_{xx} = \dfrac{\partial^2}{\partial x^2}$.

In the three-dimensional model, the demagnetizing field $h_d(m)$ is given by coupling the static Maxwell equations with the law of Faraday div $B = 0$:

$$\begin{cases} \text{curl } h_d(m) = 0 \text{ in } \mathbb{R}^3, \\ \\ \text{div } (h_d(m) + m) = 0 \text{ in } \mathbb{R}^3. \end{cases}$$

In this paper we consider a straight wire along $\mathbb{R}e_1$ with elliptical section with minor axis along $\mathbb{R}e_2$ and major axis along $\mathbb{R}e_3$, so that the one dimensional model for the demagnetizing field reads

$$h_d(m) = -m_2 e_2 - b m_3 e_3 \qquad \text{with } b \geq 1,$$

where (m_1, m_2, m_3) are the coordinates of m in $\mathbb{R}^3$. The case $b = 1$ corresponds to a wire with round section. This model for the demagnetizing field in nanowires is justified by Γ-convergence arguments in the static case and by asymptotic studies for the dynamic Landau-Lifschitz model in [5] and [6].

We first consider the case $b > 1$:

$$\begin{cases} \dfrac{\partial m}{\partial t} = -m \times H_e(m) - m \times (m \times H_e(m)) + v\partial_x m + m \times v\partial_x m, \\ \\ H_e(m) = \partial_{xx} m - m_2 e_2 - b m_3 e_3, \\ \\ |m| = 1 \text{ in } \mathbb{R}_+ \times \mathbb{R}, \end{cases} \tag{2}$$

In this case, for small values of v, the wall remains static. Indeed, we have the following proposition:

Proposition 1. *Let $b > 1$ and v such that $|v| < \sqrt{b} - 1$. We consider the following system:*

$$\begin{cases} \cos^2\theta + b\sin^2\theta = \dfrac{1}{\delta^2}, \\ (b-1)\cos\theta\sin\theta = -\dfrac{v}{\delta}. \end{cases} \tag{3}$$

Then this system admits only one solution (θ, δ) with $|\theta| < \arcsin \dfrac{1}{\sqrt{1+\sqrt{b}}}$.

In addition, we define $U_{\theta,\delta}$ by $U_{\theta,\delta}(t,x) := R_\theta(M_0(\frac{x}{\delta}))$ where

$$M_0 = \begin{pmatrix} \tanh x \\ 1/\cosh x \\ 0 \end{pmatrix} \text{ and } R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

Then $U_{\theta,\delta}$ is a static solution of System (2).

In our first result we claim that this solution is stable and asymptotically stable up to translations in the x-variable:

Theorem 1.1. *Let $b > 1$ and v such that $|v| < \sqrt{b} - 1$, let $U_{\theta,\delta}$ given by Proposition 1. Then for all $\varepsilon > 0$, there exists $\eta > 0$ such that for every m_0 in $H^2(\mathbb{R};\mathbb{R}^3)$, if m_0 satisfies the saturation constraint $|m_0| = 1$ and verifies $\|m_0 - U_{\theta,\delta}\|_{H^1} \le \eta$, if we denote by m the solution of (2) with initial data $m(0,x) = m_0(x)$ for all $x \in \mathbb{R}$, then this solution satisfies:*

$$\forall t \ge 0, \quad \|m(t,.) - U_{\theta,\delta}\|_{H^1} \le \varepsilon.$$

In addition, there exists σ_∞ such that

$$\|m(t,.) - U_{\theta,\delta}(. - \sigma_\infty)\|_{H^1} \to 0 \text{ when } t \to +\infty.$$

Remark 1. *For $|v| \ge \sqrt{b} - 1$ we observe in numerical simulations that the wall moves with a periodic velocity. In [17], a profile describing this situation is calculated, but the authors use an approximation which is not mathematically justified, so that existence and stability of any solution m for (2) in this case remain unproved.*

In the case of a nanowire with round cross-section, we have $b = 1$, so that we deal with the following system:

$$\begin{cases} \dfrac{\partial m}{\partial t} = -m \times H_e(m) - m \times (m \times H_e(m)) + v\partial_x m + m \times v\partial_x m \\ H_e(m) = \partial_{xx} m - m_2 e_2 - m_3 e_3, \\ |m| = 1. \end{cases} \tag{4}$$

For a constant applied current v, we observe a rotation and a translation of the wall profile described by the solution of (4) given by:

$$\mathrm{m}_v(t,x) = R_{-vt} M_0(x + vt),$$

We establish the stability of this solution:

Theorem 1.2. *We assume that $|v| < 2$.*
For all $\varepsilon > 0$, there exists $\eta > 0$ such that for every m_0 in the Sobolev space $H^2(\mathbb{R})$ with $|m_0| = 1$ for any $x \in \mathbb{R}$ and $\|m_0(.) - \mathrm{m}_v(0,.)\|_{H^1(\mathbb{R})} \le \eta$; if we denote by m the solution of (4) with initial data m_0, then for all $t \ge 0$ we have

$$\|m(t,.) - \mathrm{m}_v(t,.)\|_{H^1(\mathbb{R})} \le \varepsilon.$$

In addition, there exists σ_∞ and θ_∞ such that

$$\|m(t,.) - R_{\theta_\infty}\mathrm{m}_v(t, . - \sigma_\infty)\|_{H^1} \to 0 \text{ when } t \to +\infty.$$

The next theorem shows the instability of the previous solution in the case $|v| > 2$:

Theorem 1.3. *For $|v| > 2$, the solution $\mathrm{m}_v(t,x) = R_{-vt}M_0(x + vt)$ of (4) is linearly unstable.*

Remark 2. *The stability of a solution m for this Landau-Lifschitz equation with vanishing electric current but with a small applied magnetic field is treated in [6], [7] and [8]. The stability threshold for the value of the applied magnetic field is obtained in [11].*

Remark 3. *The stability results contained in this work are optimal: in the round cross section case, we establish the threshold for the value of v to obtain stability. In the elliptical case, we prove the stability for all the values of v such that the wall remains stationary. This optimality is obtained thanks to a careful study of the linearized equation around the studied profiles. This is the key point of our work.*

The present paper is organized as follows: In Section 2, after proving Proposition 1, we show the stability of the static solutions for small electric currents claimed in Theorem 1.1. The end of the paper is devoted to the case of the round section-wire. We prove the stability of moving walls for $|v| < 2$ in Section 3 and their linear instability for $|v| > 2$ in Section 4.

The framework for proving the stability theorems is the same developed in [6] and in [7]. The main difficulties are due to the following facts:

- the non linear saturation constraint $|m| = 1$,
- the invariance of the model by translation (and by rotation in the case of a round wire) so that 0 is in the spectrum of the linearized equation,
- it is not so clear that the other eigenvalues have the good sign for stability,
- the system is quasilinear so that we must use variational estimates instead of Duhamel formula.

The proofs are organized as follows.

In a first step, we transform the problem and its unknowns to deal with the stability of the profile M_0 for an equation similar to the Landau-Lifschitz equation.
In the proof of Theorem 1.1, we describe the perturbation of $U_\delta = R_\theta(M_0(\frac{x}{\delta}))$ as $m(t,x) = R_\theta(u(t,\dfrac{x}{\delta}))$, where u takes its values in S^2. Then $U_{\theta,\delta}$ is stable for System (2) if and only if the static profile M_0 is stable for a new system of unknown u. In the proof of Theorem 1.2, we describe the perturbation of $R_{-vt}M_0(x+vt)$ writing $m(t,x) = R_{-vt}u(t,x+vt)$ so that $R_{-vt}M_0(x+vt)$ is stable for 4 if and only of the static profile M_0 is stable for another new system of unknown u.

In a second step, we address the problem of the saturation constraint: we only deal with perturbation satisfying this non linear constraint. In order to do that, we describe the small perturbation u of M_0 in a moving orthonormal frame (M_0, M_1, M_2) defined by

$$M_0(x) = \begin{pmatrix} \tanh x \\ 1/\cosh x \\ 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} -1/\cosh x \\ \tanh x \\ 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

writing u as follows

$$u(t,x) = r_1(t,x)M_1(x) + r_2(t,x)M_2 + (\mu(r)+1)M_0(x), \tag{5}$$

where $\mu : B(0;1) \to \mathbb{R}$ is given by: $\mu(r_1,r_2) = \sqrt{1 - r_1^2 - r_2^2} - 1$. The new unknown $r = (r_1, r_2)$ takes its values in $\mathbb{R}^2$. After writing the Landau-Lifschitz equation with r, we obtain an equivalent formulation of u-equation where the unknown r satisfies the following non linear equation:

$$\partial_t r = \mathcal{L}r + F(x, r, \partial_x r, \partial_{xx}^2 r), \tag{6}$$

where $\mathcal{L}r$ and F represent respectively the linear part and the non linear part of this equation. We obtain that the stability of M_0 for u-equation is equivalent to the stability of zero solution for (6). We remark that we deal now with an equation taking its values in the flat space $\mathbb{R}^2$.

The key point is now the study of the spectrum of the linear operator $\mathcal{L}$. This part strongly depends on the case we deal with (see Part 2.4 for the wire with elliptical section and Part 3.4).

In both cases, zero is in the spectrum of $\mathcal{L}$. In the case of a wire with elliptical cross-section this is due to the invariance of the system with respect to translations in x so that there exists a one parameter family $\sigma \mapsto R(\sigma)$ of static solutions for (6). In addition, the eigenspace associated to zero is one dimensional. In the case of a round-section wire, the system is invariant with respect to translation in x and rotations so that there exists a two parameter family of static solutions $(\sigma,\theta) \mapsto R(\sigma,\theta)$. The null eigenspace is now two-dimensional.
The presence of zero is always a difficulty to obtain the non linear stability. To address this problem we decompose the solution r of (6) as the sum of $R(\sigma(t))$ (or $R(\sigma(t),\theta(t))$ for a wire with circular cross section) plus a perturbation W belonging to the orthogonal to the null eigenspace .

Remark 4. *This decomposition is rather classical for the study of static solution stability for semi linear parabolic equations (see [13]). This technique has also been used in [2] to demonstrate the stability of traveling waves in thin films or in [15] in the case of the radially symmetric traveling waves in reaction-diffusion equations.*

This decomposition leads us to obtain a new system of equations where the unknowns are σ and w (plus the variable θ in the circular case). Our goal is to show the stability of $(\sigma, W) = (0,0)$. The main difficulty is that our problem is quasilinear since the non linear part F depends on $\partial_{xx} r$. So we use variational methods to estimate the non linear terms.

In the following, we denote by $\langle \mid \rangle$ the inner product in the space $L^2(\mathbb{R})$ and $\|.\|_{L^2}$ the associated norm. We denote by $\cdot$ the scalar product in $\mathbb{R}^3$ and by $|.|$ the associated euclidean norm.

2 Case of a wire with elliptical cross-section

2.1 Proof of Proposition 1

We start by establishing the existence of static solutions of (2).

We recall that $U_{\theta,\delta}(x) = R_\theta(M_0(\frac{x}{\delta}))$. So, by direct calculations, denoting $y = \frac{x}{\delta}$, we obtain

$$\begin{aligned} U_{\theta,\delta} \times H_e(U_{\theta,\delta}) = \quad & R_\theta(M_1)(y) \left((b-1)\sin\theta\cos\theta \frac{1}{\cosh y} \right) \\ & + R_\theta(M_2) \left(\frac{1}{\delta^2} - \cos^2\theta - b\sin^2\theta \right) \frac{\sinh y}{\cosh^2 y} \\ \partial_x(U_{\theta,\delta}) = \quad & -\frac{1}{\delta}\frac{1}{\cosh y} R_\theta(M_1(y)) \end{aligned}$$

Writing that $U_{\theta,\delta}$ satisfies (2) if and only if

$$-U_{\theta,\delta} \times H_e(U_{\theta,\delta}) - U_{\theta,\delta} \times (U_{\theta,\delta} \times H_e(U_{\theta,\delta})) + v\partial_x U_{\theta,\delta} + vU_{\theta,\delta} \times \partial_x U_{\theta,\delta} = 0,$$

we obtain that $U_{\theta,\delta}$ satisfies (2) if and only if

$$\frac{1}{\delta^2} - \cos^2\theta - b\sin^2\theta = 0 \quad \text{ and } \quad (b-1)\sin\theta\cos\theta + \frac{v}{\delta} = 0. \tag{7}$$

We aim to prove that System (7) admits only one solution for $|v| < v_{max}$, where $v_{max} = \sqrt{b} - 1$. By elimination of δ, we obtain that v and θ are linked by the relation:

$$v = \varphi(\theta) := (1-b)\frac{\sin\theta\cos\theta}{\sqrt{1 + (b-1)\sin^2\theta}}.$$

We have

$$\varphi'(\theta) = \frac{b-1}{(1+(b-1)\sin^2\theta)^{\frac{3}{2}}}\left((b-1)\sin^4\theta + 2\sin^2\theta - 1\right).$$

We set $P(X) = (b-1)X^2 + 2X - 1$. The roots of P are $\dfrac{-1}{\sqrt{b}-1} < 0$ and $\dfrac{1}{\sqrt{b}+1} > 0$. So $\varphi'(\theta)$ is strictly negative for $\sin^2\theta \in [0, \frac{1}{\sqrt{b}+1}[$. We introduce $\theta_{max} = \arcsin\left(\dfrac{1}{\sqrt{1+\sqrt{b}}}\right)$ and $v_{max} = -\varphi(\theta_{max}) = \sqrt{b}-1$. We obtain that φ is a decreasing diffeomorphism from $]-\theta_{max}, \theta_{max}[$ to $]-v_{max}, v_{max}[$, and the proof of Proposition 1 is complete. $\square$

2.2 Proof of Theorem 1.1

2.2.1 First step: a new formulation.

We describe a perturbation m of the profile $U_{\delta,\theta}$ as:

$$m(t,x) = R_\theta(u(t, \frac{x}{\delta})),$$

where $u : \mathbb{R}^+ \times \mathbb{R} \to S^2$.
By a simple algebraic calculation, we obtain that m satisfies the Landau-Lifschitz equation (2) if and only if u satisfies the following system:

$$\begin{cases} \dfrac{\partial u}{\partial t} = -u \times h(u) - u \times (u \times h(u)) + \dfrac{v}{\delta}\left(\partial_x u + u \times \partial_x u\right), \\ h(u) = \dfrac{1}{\delta^2}\partial_{xx}u - \left(\dfrac{1}{\delta^2}u_2 - \dfrac{v}{\delta}u_3\right)e_2 - \left((b+1-\dfrac{1}{\delta^2})u_3 - \dfrac{v}{\delta}u_2\right)e_3. \end{cases} \tag{8}$$

In addition, $U_{\delta,\theta}$ is stable for (2) if and only if M_0 is stable for (8).

2.2.2 Second step: equation for the perturbations of the wall.

As in [6, 8, 11], we consider the moving frame given by $(M_0(x), M_1(x), M_2(x))$ given by

$$M_0(x) = \begin{pmatrix} \tanh x \\ 1/\cosh x \\ 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} -1/\cosh x \\ \tanh x \\ 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

If u is a small perturbation of M_0 satisfying the saturation constraint $|u| = 1$, we describe u in the mobile frame writing:

$$u(t,x) = r_1(t,x)M_1(x) + r_2(t,x)M_2 + (\mu(r)+1)M_0(x), \tag{9}$$

where $r = (r_1, r_2) \in \mathbb{R}^2$ and $\mu(r) = \sqrt{1 - r_1^2 - r_2^2} - 1$. The study of small perturbations of M_0 allows us to assume that $\|u - M_0\|_{L^\infty} \leq \dfrac{1}{2}$.

We plug (9) in (8) and we obtain that if u satisfies (8) satisfies the following equation:

$$\partial_t r = \frac{1}{\delta^2}JNr + F(x, r, \partial_x r, \partial_{xx} r). \tag{10}$$

In the linear part $\dfrac{1}{\delta^2}JN$,

$$J = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} L & -\delta v l^* \\ -\delta v l & L + \alpha \end{pmatrix},$$

where $L=-\partial_{xx}+2\tanh^2 x-1$, $l=\partial_x+\tanh x$ and $\alpha=\delta^2(b+1)-2$.

The non linear part $F:\mathbb{R}\times B(0,1)\times\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}^2$ is defined by

$$F(x,r,\partial_x r,\partial_{xx}r)=A(r)\partial_{xx}r+B(r)(\partial_x r,\partial_x r)+C(x,r)(\partial_x r)+D(r)(\partial_x r)+E(x,r),\qquad(11)$$

where

- $A\in C^\infty(B(0,1);\mathcal{M}_2(\mathbb{R}))$ ($\mathcal{M}_2(\mathbb{R})$ is the set of the real 2×2 matrices):

$$A(r)\xi=\frac{1}{\delta^2}\begin{pmatrix}-r_1^2 & \mu-r_1r_2\\ -\mu-r_1r_2 & -r_2^2\end{pmatrix}\xi+\frac{1}{\delta^2}\begin{pmatrix}-r_2-r_1(1+\mu(r))\\ r_1-r_2(1+\mu(r))\end{pmatrix}\mu'(r)(\xi),$$

- $B\in C^\infty(B(0,1);\mathcal{L}_2(\mathbb{R}^2))$ ($\mathcal{L}_2(\mathbb{R}^2)$ is the set of the bilinear functions defined on $\mathbb{R}^2\times\mathbb{R}^2$ with values in$\mathbb{R}^2$) :

$$B(r)(\xi,\xi)=\frac{1}{\delta^2}\begin{pmatrix}-(\mu+1)r_1-r_2\\ -(\mu+1)r_2+r_1\end{pmatrix}\mu''(r)(\xi,\xi),$$

- $C\in C^\infty(\mathbb{R}\times B(0,1);\mathcal{M}_2(\mathbb{R}))$:

$$C(x,r)(\xi)=\frac{2}{\delta^2\cosh x}\begin{pmatrix}-r_1(1+\mu(r))-r_2\\ r_1-r_2(1+\mu(r))\end{pmatrix}\xi_1+\frac{2}{\delta^2\cosh x}\begin{pmatrix}(r_1)^2\\ r_1r_2\end{pmatrix}\mu'(r)(\xi)$$

- $D\in C^\infty(\mathbb{R};\mathcal{M}_2(\mathbb{R}))$:

$$D(r)(\xi)=\frac{v}{\delta}\mu(r)\begin{pmatrix}-\xi_2\\ \xi_1\end{pmatrix}+\frac{v}{\delta}\begin{pmatrix}r_2\\ -r_1\end{pmatrix}\mu'(r)(\xi)$$

- $E\in C^\infty(\mathbb{R}\times B(0,1);\mathbb{R}^2)$:

$$\begin{aligned}E(x,r)=\;&\frac{2\sinh x}{\delta^2\cosh^2 x}\begin{pmatrix}r_1r_2+r_1^2(1+\mu(r))\\ -(r_1)^2+r_1r_2(1+\mu(r))\end{pmatrix}\\
&+\frac{v}{\delta\cosh x}\begin{pmatrix}-(r_2)^2+\mu(r)+(\mu(r))^2-r_1r_2(1+2\mu(r))\\ r_1r_2-2(r_2)^2(1+\mu(r))-(r_1)^2-\mu(r)(1+\mu(r))\end{pmatrix}\\
&+\frac{v}{\delta}\tanh x\begin{pmatrix}r_1\mu(r)-2(r_1)^2r_2\\ -\mu(r)r_2-2r_1(r_2)^2\end{pmatrix}+\frac{2}{\delta^2\cosh^2 x}\begin{pmatrix}\mu(r)(r_1(2+\mu(r))+r_2)\\ \mu(r)(-r_1+r_2(1+\mu(r)))\end{pmatrix}\\
&+\begin{pmatrix}(b+1-\frac{1}{\delta^2})(r_2\mu(r)+r_1(r_2)^2)+\frac{1}{\delta^2}(r_1)^3\\ \frac{1}{\delta^2}(r_1\mu(r)+(r_1)^2r_2)+(b+1-\frac{1}{\delta^2})(r_2)^3\end{pmatrix}\end{aligned}$$

A simple projection of (8) on the mobile frame $(M_1(x),M_2)$ ensures that u satisfies (8) implies that r satisfies (10). The reverse is proved in detail in [6] using the fact that, since (10) preserves the saturation constraint $|u|=1$, if u satisfies the projections of (6) on M_1 and M_2, then it satisfies (8). In addition, we remark that u is stable for (8) if and only if 0 is stable for (10).

2.3 Invariance by translation and new formulation

The modified Landau-Lifschitz system (8) is invariant by translation in the x-variable so that $x \mapsto M_0(x-s)$ is a static solution of (8) for all $s \in \mathbb{R}$. By projection on the mobile frame $(M_0(x), M_1(x), M_2)$, we obtain a one parameter family $R(s)$ of static solutions for (10) defined by:

$$R(s)(x) = \begin{pmatrix} M_0(x-s).M_1(x) \\ \\ M_0(x-s).M_2 \end{pmatrix} = \begin{pmatrix} \rho(s)(x) \\ \\ 0 \end{pmatrix},$$

where $\rho(s)(x) = \dfrac{\tanh x}{\cosh(x-s)} - \dfrac{\tanh(x-s)}{\cosh x}$.

The existence of this one parameter family of solutions implies that 0 is an eigenvalue of the linearized of (10) around zero. Indeed we have:

$$N \begin{pmatrix} \dfrac{1}{\cosh x} \\ 0 \end{pmatrix} = 0.$$

This fact obstructs immediate getting of the stability result. To overcome this difficult, we isolate the translation writing the solution r of (10) on the form:

$$r(t,x) = R(\sigma(t))(x) + W(t,x) \tag{12}$$

where $\sigma : \mathbb{R}^+ \to \mathbb{R}$ and $W = \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^2$, with $\langle W_1 | \frac{1}{\cosh x} \rangle = 0$.

In a neighborhood of zero, this is a valid system of coordinates, as it is claimed in the following proposition:

Proposition 2. *There exists $\delta_0 > 0$ such that for all $k \geq 1$, for all $r \in H^k(\mathbb{R}, \mathbb{R}^2)$ satisfying $\|r\|_{L^\infty} \leq \delta_0$, there exists a unique couple $(\sigma, W) \in \mathbb{R} \times H^k(\mathbb{R}; \mathbb{R}^2)$ such that:*

$$r(x) = R(\sigma)(x) + W(x)$$

with $\langle W_1 | \dfrac{1}{\cosh x} \rangle = 0$.

Proof. Let us assume that r writes: $r = R(\sigma)(x) + W(x)$ where $\langle W_1 | \dfrac{1}{\cosh x} \rangle = 0$. By taking the scalar product with $\begin{pmatrix} \dfrac{1}{2\cosh x} \\ 0 \end{pmatrix}$, we obtain that

$$\langle r_1 | \frac{1}{2\cosh x} \rangle = \psi(\sigma) \quad \text{where } \psi \text{ is defined by } \psi(s) = \frac{1}{2} \int_{x \in \mathbb{R}} \rho(s)(x) \cdot \frac{1}{\cosh x} dx.$$

The map ψ is smooth and we have $\psi(0) = 0$ and $\psi'(0) = 1$. So, in a neighborhood of zero, there exists $\delta_0 > 0$ such that ψ is a C^∞-diffeomorphism from $]-\delta_0, \delta_0[$ to a neighborhood of $\psi(0) = 0$. So, σ can be obtained by: $\sigma(t) = \psi^{-1}(\langle r_1 | \frac{1}{2\cosh x} \rangle)$.

By subtraction, we define W by $W(x) = r(x) - R(\sigma)(x)$, and by construction we have that $\langle W_1 | \frac{1}{\cosh x} \rangle = 0$.

□

We plug (12) in (8) and we obtain that

$$\partial_s R(\sigma)\sigma' + \partial_t W = \frac{1}{\delta^2} JNR(\sigma) + \frac{1}{\delta^2} JNW + F(x, R(\sigma) + W, \partial_x R(\sigma) + \partial_x W, \partial_{xx} R(\sigma) + \partial_{xx} W). \tag{13}$$

Since, for a fixed s, $R(s)$ is a static solution of (8) we obtain that

$$\frac{1}{\delta^2} JNR(\sigma) + F(x, R(\sigma), \partial_x R(\sigma), \partial_{xx} R(\sigma)) = 0$$

so we obtain from (13) that

$$\partial_s R(\sigma)\sigma' + \partial_t W = \frac{1}{\delta^2} JNW + G(x,\sigma,W,\partial_x W,\partial_{xx}W), \tag{14}$$

where

$$\begin{aligned} G(x,\sigma,W,\partial_x W,\partial_{xx}W) = \quad & F(x,R(\sigma)+W,\partial_x R(\sigma)+\partial_x W,\partial_{xx}R(\sigma)+\partial_{xx}W) \\ & -F(x,R(\sigma),\partial_x R(\sigma),\partial_{xx}R(\sigma)). \end{aligned}$$

In order to obtain the equation satisfied by σ, we take the inner product of (14) with $\kappa_1 = \begin{pmatrix} \frac{1}{2\cosh x} \\ 0 \end{pmatrix}$.

We first remark that $\langle \partial_t W | \kappa_1 \rangle = 0$ since $W_1 \in (\frac{1}{\cosh x})^{\perp}$.
In addition, $\langle \frac{1}{\delta^2} JNW | \kappa_1 \rangle = \lambda(W)$ where λ is defined by

$$\lambda(W) = \frac{v}{2\delta}\langle \frac{1}{\cosh x} | W_1 \rangle - \frac{\alpha}{2\delta^2}\langle \frac{1}{\cosh x} | W_2 \rangle. \tag{15}$$

We denote by $g(s)$ the map defined by

$$g(s) = \int_{x\in\mathbb{R}} \frac{\partial \rho}{\partial s}(s)(x) \frac{1}{\cosh x} dx.$$

We remark that $g(0) = 1$ and g is smooth so that $s \mapsto \dfrac{1}{g(s)}$ is smooth in a neighborhood of zero.
We denote:

$$\widetilde{G}(\sigma, W) = \langle G(x,\sigma,W,\partial_x W,\partial_{xx}W) | \kappa_1 \rangle. \tag{16}$$

Therefore we have:

$$\frac{d\sigma}{dt} = \frac{1}{g(\sigma)}\left(\lambda(W) + \widetilde{G}(\sigma, W)\right). \tag{17}$$

By subtraction we obtain the following system for W:

$$\partial_t W = \frac{1}{\delta^2} JNW - \frac{1}{g(\sigma)}\left(\lambda(W) + \widetilde{G}(\sigma,W)\right)\partial_s R(\sigma) + G(x,\sigma,W,\partial_x W,\partial_{xx}W). \tag{18}$$

2.4 Estimate for the linear part

We recall that N is given by

$$N = \begin{pmatrix} L & -\delta v l^* \\ -\delta v l & L+\alpha \end{pmatrix}.$$

The operator $L = -\partial_{xx} + 2(\tanh x)^2 - 1$ arising in the formulation of N is a self-adjoint operator acting on $H^2(\mathbb{R}^3)$. As it is proved in [6] we have the following proposition:

Proposition 3. *The essential spectrum of the self-adjoint operator L is $[1, +\infty[$ and zero is its unique eigenvalue which eigenspace is generated by $\dfrac{1}{\cosh x}$.*

Proof. Since L is a compact perturbation of the operator $-\partial_{xx} + 1$ then its essential spectrum is $[1, +\infty[$ (see [12]). Furthermore, we can see that $L = l^* l$, then L is positive and 0 is a simple eigenvalue associated to the eigenvector $\dfrac{1}{\cosh x}$. Now, let us assume that λ is an eigenvalue of L and w is an eigenvector associated to λ, that is $Lw = \lambda w$. Applying of the operator l on this equation and using the relation $l^* l = -\partial_{xx} + 1$ shows that λ is an eigenvalue for $-\partial_{xx} + 1$ associated to the eigenvector lw, which ensures that $lw = 0$ so that $\lambda = 0$. $\square$

From this result we obtain the following corollary:

Corollary 1. *For al $w \in H^2(\mathbb{R})$ satisfying $\langle w | \frac{1}{\cosh x} \rangle = 0$, we have:*

$$\|w\|_{L^2} \leq \sqrt{\langle Lw|w \rangle} \leq \|LW\|_{L^2}^2.$$

In addition there exists constants c_1 and c_2 such that for all $w \in \left(\frac{1}{\cosh x}\right)^{\perp}$,

$$c_1 \|w\|_{H^1} \leq \sqrt{\langle Lw|w \rangle} \leq c_2 \|w\|_{H^1}$$

and

$$c_1 \|w\|_{H^2} \leq \|Lw\|_{L^2} \leq c_2 \|w\|_{H^2}.$$

We prove now that N is coercive on $(Ker\, L)^{\perp} \times H^2(\mathbb{R})$.

Proposition 4. *There exists a constant $c_v > 0$ such that for all $W \in H^2(\mathbb{R}; \mathbb{R}^2)$ with $\langle W_1 | \frac{1}{\cosh x} \rangle = 0$,*

$$\langle NW|W \rangle \geq c_v \|W\|_{L^2}^2.$$

Proof. The key point is the following estimate for $\langle NW|W \rangle$:

$$\begin{aligned}
\langle NW|W \rangle = & \ \langle LW_1|W_1 \rangle - \delta v \langle l^* W_2|W_1 \rangle - \delta v \langle lW_1|W_2 \rangle + \langle LW_2|W_2 \rangle + \alpha \langle W_2|W_2 \rangle \\
\geq & \ \langle lW_1|lW_1 \rangle - 2\delta v \langle W_2|lW_1 \rangle + \alpha \langle W_2|W_2 \rangle \quad \text{since } L = l^* \circ l \text{ is positive} \\
\geq & \ \|lW_1\|_{L^2}^2 - 2\delta |v| \|W_2\|_{L^2} \|lW_1\|_{L^2} + \|W_2\|_{L^2}^2
\end{aligned}$$

We have then:

$$\langle NW|W \rangle \geq \begin{pmatrix} \|lW_1\|_{L^2} & \|W_2\|_{L^2} \end{pmatrix} \begin{pmatrix} 1 & -\delta v \\ -\delta v & \alpha \end{pmatrix} \begin{pmatrix} \|lW_1\|_{L^2} \\ \|W_2\|_{L^2} \end{pmatrix}$$

We introduce the matrix $\mathcal{M}_v = \begin{pmatrix} 1 & -\delta v \\ -\delta v & \alpha \end{pmatrix}$. We denote by c_v the smallest eigenvalue of $\mathcal{M}_v$. We have:

$$\langle NW|W \rangle \geq c_v \left(\|lW_1\|_{L^2}^2 + \|W_2\|_{L^2}^2 \right) \geq c_v \left(\|W_1\|_{L^2}^2 + \|W_2\|_{L^2}^2 \right)$$

since $W_1 \in (\frac{1}{\cosh x})^{\perp}$. It remains to prove that $c_v > 0$

We have

$$c_v = \frac{1 + \alpha - \sqrt{(1-\alpha)^2 + 4\delta^2 v^2}}{2}$$

so that $c_v > 0$ if and only if $\alpha > \delta^2 v^2$.

Using (7), we remark that $\alpha - \delta^2 v^2 = \dfrac{\delta^2 (b-1)}{\cos^2 \theta + b \sin^2 \theta} (1 - 2\sin^2 \theta - (b-1) \sin^4 \theta)$ and this quantity is non negative since $\sin^2 \theta$ remains between the roots of the polynomial map $1 - 2X - (b-1)X^2$ (see subsection 2.1).

□

We deduce from the previous Proposition the following equivalence of norms:

Corollary 2. *There exists c_1 and c_2 with $0 < c_1 < c_2$ such that for all $W \in H^2(\mathbb{R}; \mathbb{R}^2)$ with $\langle W_1 | \frac{1}{\cosh x} \rangle = 0$,*

$$c_1 \|W\|_{H^1(\mathbb{R})} \leq \sqrt{\langle NW|W \rangle} \leq c_2 \|W\|_{H^1(\mathbb{R})}$$

$$c_1 \|W\|_{H^2(\mathbb{R})} \leq \|NW\| \leq c_2 \|W\|_{H^1(\mathbb{R})}$$

To conclude this part, we estimate the operator λ arising in (17) and (18):

Proposition 5. *There exists C such that for all $W \in H^1(\mathbb{R})$,*

$$|\lambda(W)| \leq C \|W\|_{L^2(\mathbb{R})}.$$

Proof. We deduce this estimate from the expression of λ (see (15)). □

2.5 Estimate for the non linear terms

We recall that the non linear contribution $F(x, r, \partial_x r, \partial_{xx} r)$ in Equation (10) is detailed in (11). We estimate the nonlinear functions A, B, C, D and E appearing in this nonlinear term. Since $\mu(r) = O(|r|^2)$, by straightforward calculations, we obtain the following proposition:

Proposition 6. *There exists a constant C such that for $r \in B(0, \frac{1}{2})$ and for $x \in \mathbb{R}$,*

- $|A(r)| \leq C|r|^2$ *and* $|A'(r)| \leq C|r|$,
- $|B(r)| \leq C|r|$ *and* $|B'(r)| \leq C$,
- $|C(x,r)| \leq \dfrac{C}{\cosh x}\,|r|$ *and* $|\dfrac{\partial}{\partial r} C(x,r)| \leq \dfrac{C}{\cosh x}$,
- $|D(r)| \leq C|r|^2$
- $|E(x,r)| \leq C|r|^2$ *and* $|\dfrac{\partial}{\partial r} E(x,r)| \leq C|r|$,

Now, the non linear term G is deduced from F writing

$$\begin{aligned} G(x, \sigma, W, \partial_x W, \partial_{xx} W) = \quad & F(x, R(\sigma) + W, \partial_x R(\sigma) + \partial_x W, \partial_{xx} R(\sigma) + \partial_{xx} W) \\ & - F(x, R(\sigma), \partial_x R(\sigma), \partial_{xx} R(\sigma)). \end{aligned} \tag{19}$$

We detail the non linear term G writing the Fundamental Theorem of the Analysis between R_σ and $R_\sigma + W$ writing $G = G_1 + \ldots + G_5$ with:

- $G_1 = A(R(\sigma) + W)\partial_{xx} W + \tilde{A}(R(\sigma), W)(W)(\partial_{xx} R(\sigma))$, with $\tilde{A}(u,v) = \displaystyle\int_0^1 A'(u + sv)ds$,
- $G_2 = B(R(\sigma)+W)(2\partial_x R(\sigma)+\partial_x W, \partial_x W)+\tilde{B}(R(\sigma), W)(W)(\partial_x R(\sigma), \partial_x R(\sigma))$, with $\tilde{B}(u,v) = \displaystyle\int_0^1 B'(u + sv)ds$,
- $G_3 = C(x, R(\sigma) + W)(\partial_x W) + \tilde{C}(x, R(\sigma), W)(W)(\partial_x R(\sigma))$ with $\tilde{C}(x,u,v) = \displaystyle\int_0^1 \frac{\partial}{\partial r} C(x, u + sv)ds$,
- $G_4 = D(R(\sigma) + W)(\partial_x W) + \tilde{D}(R(\sigma), W)(W)(\partial_x R(\sigma))$ with $\tilde{D}(u,v) = \displaystyle\int_0^1 \frac{\partial}{\partial r} D(u + sv)ds$,
- $G_5 = \tilde{E}(x, R(\sigma), W)(W)$ with $\tilde{E}(x,u,v) = \displaystyle\int_0^1 \frac{\partial}{\partial r} E(x, u + sv)ds$,

We remark that there exists a constant C such that

$$|R(\sigma) + W| \leq C(|\sigma| + |W|). \tag{20}$$

From the properties detailed in Proposition 6, we obtain that there exists $\eta_0 > 0$, there exists a constant K such that if $|\sigma| + \|W\|_{L^\infty} \leq \eta_0$

$$\begin{aligned} &\|G_1\|_{L^2} \leq K(|\sigma| + \|W\|_{L^\infty})(\|\partial_{xx} W\|_{L^2} + \|W\|_{L^2}) \\ &\|G_2\|_{L^2} \leq K\left(|\sigma|\|\partial_x W\|_{L^2} + \|\partial_x W\|_{L^4}^2 + \|W\|_{L^2}|\sigma|\right) \\ &\|G_3\|_{L^2} \leq K\left((|\sigma| + \|W\|_{L^\infty})\|\partial_x W\|_{L^2} + |\sigma|\|W\|_{L^2}\right) \\ &\|G_4\|_{L^2} \leq K\left((|\sigma| + \|W\|_{L^\infty})\|\partial_x W\|_{L^2} + |\sigma|\|W\|_{L^2}\right) \\ &\|G_5\|_{L^2} \leq K\left((|\sigma| + \|W\|_{L^\infty})\|W\|_{L^2}\right). \end{aligned} \tag{21}$$

Using that $\|W\|_{L^\infty} \leq C\|W\|_{H^1}$ by Sobolev injection, using the Gagliardo-Nirenberg inequality $\|\partial_x u\|_{L^4}^2 \leq C\|u\|_{L^\infty}\|\partial_{xx}u\|_{L^2}$ for the estimate of G_2, we obtain that if $|\sigma| + \|W\|_{L^\infty} \leq \eta_0$,

$$\|G\|_{L^2} \leq C|\sigma|\|W\|_{H^2} + C\|W\|_{H^1}\|W\|_{H^2}. \tag{22}$$

Let us now estimate $\widetilde{G}$ defined by (16). We recall that $\kappa_1 = \begin{pmatrix} \frac{1}{2\cosh x} \\ 0 \end{pmatrix}$. We estimate successively each term $\langle G_i|\kappa_1\rangle$.

- $\langle G_1|\kappa_1\rangle = \langle A(R(\sigma)+W)\partial_{xx}W|\kappa_1\rangle + \langle \tilde{A}(R(\sigma),W)(W)(\partial_{xx}R(\sigma))|\kappa_1\rangle$

  On the one hand, the first term satisfies

$$\begin{aligned}
\langle A(R(\sigma)+W)\partial_{xx}W|\kappa_1\rangle = & \ \langle \partial_{xx}W|A(R(\sigma)+W)^*\kappa_1\rangle \\
= & \ -\langle \partial_x W|\partial_x(A(R(\sigma)+W)^*\kappa_1)\rangle \\
= & \ -\langle \partial_x W|(A'(R(\sigma)+W)(\partial_x W))^*\kappa_1\rangle \\
& -\langle \partial_x W|(A'(R(\sigma)+W)(\partial_x R(\sigma))^*\kappa_1\rangle \\
& -\langle \partial_x W|A(R(\sigma)+W)^*\partial_x\kappa_1)\rangle
\end{aligned}$$

  so

$$\begin{aligned}
\left|\langle A(R(\sigma)+W)\partial_{xx}W|\kappa_1\rangle\right| \leq & \ \|\partial_x W\|_{L^2}^2\|A'(R(\sigma)+W)\|_{L^\infty}\|\kappa_1\|_{L^\infty} \\
& +\|\partial_x W\|_{L^2}\|A'(R(\sigma)+W)\|_{L^\infty}\|\partial_x R(\sigma)\|_{L^\infty}\|\kappa_1\|_{L^2} \\
& +\|\partial_x W\|_{L^2}\|A(R(\sigma)+W)\|_{L^\infty}\|\partial_x\kappa_1\|_{L^2}
\end{aligned}$$

  Therefore, if $|\sigma| + \|W\|_{L^\infty} \leq \eta_0$

$$|\langle A(R(\sigma)+W)\partial_{xx}W|\kappa_1\rangle| \leq C\|\partial_x W\|_{L^2}^2 + C\|\partial_x W\|_{L^2}$$

  On the other hand,

$$\left|\langle \tilde{A}(R(\sigma),W)(W)(\partial_{xx}R(\sigma))|\kappa_1\rangle\right| \leq \|\tilde{A}(R(\sigma),W)\|_{L^\infty}\|W\|_{L^2}\|\partial_{xx}R(\sigma)\|_{L^\infty}\|\kappa_1\|_{L^2}$$

  so, if $|\sigma| + \|W\|_{L^\infty} \leq \eta_0$

$$\left|\langle \tilde{A}(R(\sigma),W)(W)(\partial_{xx}R(\sigma))|\kappa_1\rangle\right| \leq \|W\|_{L^2}.$$

  Therefore, if $|\sigma| + \|W\|_{L^\infty} \leq \eta_0$

$$|\langle G_1|\kappa_1\rangle| \leq C\|W\|_{H^1} + C\|W\|_{H^1}^2.$$

- Concerning $\langle G_2|\kappa_1\rangle$, we have:

$$\begin{aligned}
\left|\langle G_2|\kappa_1\rangle\right| \leq & \ 2\|B(R(\sigma)+W)\|_{L^\infty}\|\partial_x R(\sigma)\|_{L^\infty}\|\partial_x W\|_{L^2}\|\kappa_1\|_{L^2} \\
& +\|B(R(\sigma)+W)\|_{L^\infty}\|\partial_x W\|_{L^2}^2\|\kappa_1\|_{L^\infty} \\
& +\|\tilde{B}(R(\sigma),W)\|_{L^\infty}\|W\|_{L^2}\|\partial_x R(\sigma)\|_{L^\infty}^2\|\kappa_1\|_{L^2}
\end{aligned}$$

  so that if $|\sigma| + \|W\|_{L^\infty} \leq \eta_0$,

$$|\langle G_2|\kappa_1\rangle| \leq C\|W\|_{H^1} + C\|W\|_{H^1}^2.$$

- From the last three estimates in (21) we obtain that if $|\sigma| + \|W\|_{L^\infty} \leq \eta_0$,

$$|\langle G_3 + G_4 + G_5 | \kappa_1 \rangle| \leq C\|W\|_{H^1}.$$

Therefore, there exists a constant C such that if $|\sigma| + \|W\|_{L^\infty} \leq \eta_0$,

$$\left|\widetilde{G}\right| \leq C\|W\|_{H^1} + C\|W\|_{H^1}^2. \tag{23}$$

2.6 End of the proof

We take the inner product of (18) with NW. Since N is a self-adjoint operator we obtain that:

$$\frac{d}{dt}\langle W|NW\rangle + \frac{1}{\delta^2}\|NW\|_{L^2}^2 = -\frac{1}{g(\sigma)}\left(\lambda(W) + \widetilde{G}\right)\langle \partial_s R(\sigma)|NW\rangle + \langle G|NW\rangle.$$

We remark that

$$\partial_s R(s) = \begin{pmatrix} \frac{1}{\cosh x} \\ 0 \end{pmatrix} + s\begin{pmatrix} \tau(s)(x) \\ 0 \end{pmatrix}$$

where τ is smooth and uniformly bounded in $L^2(\mathbb{R}_x)$ for s in a neighborhood of zero.
In addition, we remark that

$$\left\langle \begin{pmatrix} \frac{1}{\cosh x} \\ 0 \end{pmatrix} |NW\right\rangle = \left\langle \frac{1}{\cosh x}|LW_1 - \delta v l^* W_2\right\rangle = \left\langle L\frac{1}{\cosh x}|W_1\right\rangle - \delta v\left\langle l\frac{1}{\cosh x}|W_2\right\rangle = 0$$

since $L\frac{1}{\cosh x} = l\frac{1}{\cosh x} = 0$. Therefore we obtain that

$$\begin{aligned} \frac{d}{dt}\langle W|NW\rangle + \frac{1}{\delta^2}\|NW\|_{L^2}^2 = \quad & -\sigma\frac{1}{g(\sigma)}\left(\lambda(W) + \widetilde{G}\right)\langle \tau(\sigma)(x)|NW\rangle + \langle G|NW\rangle \\ \leq \quad & C|\sigma|\left(|\lambda(W)| + |\widetilde{G}|\right)\|\tau(\sigma)\|_{L^2}\|NW\|_{L^2} + \|G\|_{L^2}\|NW\|_{L^2}. \end{aligned}$$

Using the Sobolev embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$, using the equivalence of norms claimed in Corollary 2, using Proposition 5 and the estimates (22) and (23), we obtain that there exists $\eta_1 > 0$, there exists a constant K such that if $|\sigma| \leq \eta_1$ and $\langle W|NW\rangle^{\frac{1}{2}} \leq \eta_1$,

$$\frac{d}{dt}\langle W|NW\rangle + \frac{1}{\delta^2}\|NW\|_{L^2}^2 \leq K\left(|\sigma| + \langle W|NW\rangle^{\frac{1}{2}}\right)\|NW\|_{L^2}^2$$

therefore

$$\frac{d}{dt}\langle W|NW\rangle + \|NW\|_{L^2}^2\left(\frac{1}{\delta^2} - K\left(|\sigma| + \langle W|NW\rangle^{\frac{1}{2}}\right)\right) \leq 0.$$

So while $|\sigma| \leq \min(\frac{1}{3\delta^2 K}, \eta_1)$, while $\langle W|NW\rangle^{\frac{1}{2}} \leq \min(\frac{1}{3\delta^2 K}, \eta_1)$, we have:

$$\frac{d}{dt}\langle W|NW\rangle + \|NW\|_{L^2}^2\frac{1}{3\delta^2} \leq 0.$$

Using Corollary 2, we have $\|NW\|_{L^2}^2 \geq (c_1)^2\|W\|_{H^2}^2 \geq (c_1)^2\|W\|_{H^1}^2 \geq \left(\frac{c_1}{c_2}\right)^2\langle W|NW\rangle$, therefore, while $|\sigma| \leq \min(\frac{1}{3\delta^2 K}, \eta_1)$, while $\langle W|NW\rangle^{\frac{1}{2}} \leq \min(\frac{1}{3\delta^2 K}, \eta_1)$, we have:

$$\frac{d}{dt}\langle W|NW\rangle + \frac{(c_1)^2}{3(c_2)^2\delta^2}\langle W|NW\rangle \leq 0,$$

so that by comparison lemma:

$$\langle W|NW\rangle(t) \leq \langle W|NW\rangle(0)\exp\left(-\frac{(c_1)^2 t}{3(c_2)^2\delta^2}\right).$$

Now, using (17), Proposition 5 and Estimate (23), we obtain that while $|\sigma| \leq \min(\frac{1}{3\delta^2 K}, \eta_1)$, while $\langle W|NW\rangle^{\frac{1}{2}} \leq \min(\frac{1}{3\delta^2 K}, \eta_1)$, we have:

$$\left|\frac{d\sigma}{dt}\right| \leq C\|W\|_{H^1} \leq \frac{C}{c_1}\sqrt{\langle W|NW\rangle} \leq \frac{C}{c_1}\sqrt{\langle W|NW\rangle(0)}\exp\left(-\frac{(c_1)^2 t}{6(c_2)^2\delta^2}\right).$$

So by integration, we obtain that while $|\sigma| \leq \min(\frac{1}{3\delta^2 K}, \eta_1)$, while $\langle W|NW\rangle^{\frac{1}{2}} \leq \min(\frac{1}{3\delta^2 K}, \eta_1)$, we have:

$$|\sigma(t)| \leq |\sigma(0)| + \frac{C}{c_1}\sqrt{\langle W|NW\rangle(0)}\frac{6(c_2)^2\delta^2}{(c_1)^2}.$$

Therefore if $\sigma(0)$ and $\langle W|NW\rangle(0)$ are small enough, for all $t \geq 0$, $|\sigma| \leq \min(\frac{1}{3\delta^2 K}, \eta_1)$ and $\langle W|NW\rangle^{\frac{1}{2}} \leq \min(\frac{1}{3\delta^2 K}, \eta_1)$, so that all the previous inequalities hold. In particular, $W \to 0$ in $H^1(\mathbb{R})$ and since $\dfrac{d\sigma}{dt}$ is integrable on $\mathbb{R}^+$, σ tends to a limit σ_∞ when t tends to $+\infty$.

This conclude the proof of Theorem 1.1. □

3 Case of a round wire.

In this part, we deal with the following model:

$$\begin{cases} m : \mathbb{R}_t^+ \times \mathbb{R}_x \longrightarrow S^2, \\ \dfrac{\partial m}{\partial t} = -m \times h_e(m) - m \times (m \times h_e(m)) + v\partial_x m + vm \times \partial_x m \\ h_e(m) = \partial_{xx} m - m_2 e_2 - m_3 e_3. \end{cases} \tag{24}$$

where $h_e(m)$ is the effective field, v is a transport term, (e_1, e_2, e_3) is the canonical basis of $\mathbb{R}^3$ and the nanowire is modeled by the axis $\mathbb{R}e_1$.
We remark that in this case we can replace the effective field $h_e(m)$ by $\tilde{h}_e(m) = \partial_{xx} m + m_1 e_1$ since it only appears $m \times h_e(m)$ in the equation.

For a constant applied courant v, we consider the solution m_v modeling one-wall configurations: by

$$\mathrm{m}_v(t, x) = R_{-vt} M_0(x + vt).$$

We aim to prove the stability of this solution for $|v| < 2$ claimed in Theorem 1.2. We will follow the same methodology of the previous section.

3.1 A new equation

As we did in the previous section, we define a new variable u by the relation

$$m(t, x) = R_{-vt} u(t, x + vt).$$

Since the saturation constraint $|m(t, x)| = 1$ is satisfied by m for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, then u satisfies also this constraint because the matrix R_{-vt} preserves the euclidean norm. By a simple calculation, we obtain that m is solution of (24) if and only if u satisfies the following system

$$\begin{cases} \dfrac{\partial u}{\partial t} = -u \times h(u) - u \times (u \times h(u)) + vu \times \partial_x u - vu \times e_1 \\ h(u) = \partial_{xx} u + u_1 e_1, \end{cases} \tag{25}$$

and m_v is stable for (24) if and only if the static profile M_0 is stable for (25).

3.2 Mobile frame.

We consider u as a little perturbation of M_0. Let us write u in the moving frame (M_0, M_1, M_2) where the physical constraint $|u| = 1$ is automatically satisfied:

$$u(t,x) = r_1(t,x)M_1(x) + r_2(t,x)M_2 + (\mu(r)+1)M_0(x),$$

where μ is a smooth map defined in the first section.
After a rather long algebraic calculation, we obtain the following result:

Proposition 7. *The function u satisfies (25) if and only if r satisfies*

$$\frac{\partial r}{\partial t} = \mathcal{L}r + F_0(x, r, \partial_x r, \partial_{xx} r), \tag{26}$$

where the linear operator $\mathcal{L}$ is defined by

$$\mathcal{L} = \begin{pmatrix} -L & -L - vl \\ L + vl & -L \end{pmatrix}$$

with $L = -\partial_{xx} + (2\tanh^2 x - 1)$ and $l = \partial_x + \tanh x$, and where F_0 is the non linear part of (26) given by:

$$F_0(x, r, \partial_x r, \partial_{xx} r) = A_0(r)\partial_{xx} r + B_0(x,r)\partial_x r + C_0(r)(\partial_x r, \partial_x r) + D_0(x,r)$$

where

- $A_0 \in C^\infty(B(0,1); \mathcal{M}_2(\mathbb{R}))$:

$$A_0(r)(\xi) = \begin{pmatrix} -r_1^2 & \mu - r_1 r_2 \\ -\mu - r_1 r_2 & -r_2^2 \end{pmatrix} \xi + \begin{pmatrix} -r_2 - r_1(1+\mu(r)) \\ r_1 - r_2(1+\mu(r)) \end{pmatrix} \mu'(r)(\xi),$$

- $B_0 \in C^\infty(\mathbb{R} \times B(0,1); \mathcal{M}_2(\mathbb{R}))$:

$$B_0(x,\xi) = \frac{2}{\cosh x} \begin{pmatrix} -r_1(1+\mu(r)) - r_2 \\ r_1 - r_2(1+\mu(r)) \end{pmatrix} \xi_1 + v\mu(r) \begin{pmatrix} -\xi_2 \\ \xi_1 \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{2}{\cosh x}((r_1)^2 - 1) + v r_2 \\ \frac{2}{\cosh x}(1 + \mu(r) + r_1 r_2) - v r_1 \end{pmatrix} \mu'(r)(\xi)$$

- $C_0 \in C^\infty(\mathbb{R}; \mathcal{L}_2(\mathbb{R}^3))$

$$C_0(r)(\xi,\xi) = \begin{pmatrix} -r_2 - r_1(1+\mu(r)) \\ r_1 - r_2(1+\mu(r)) \end{pmatrix} \mu''(r)(\xi,\xi)$$

- $D_0 \in C^\infty(\mathbb{R} \times B(0,1); \mathbb{R}^2)$*:*

$$D_0(x,r) = \frac{\sinh x}{\cosh^2 x} \begin{pmatrix} 2r_1 r_2 + 2(r_1)^2(1+\mu(r)) \\ -2(r_1)^2 + 2r_1 r_2(1+\mu(r)) \end{pmatrix} + \frac{v}{coshx} \begin{pmatrix} r_1 r_2 \\ -(r_1)^2 - \mu(r)(1+\mu(r)) \end{pmatrix}$$

$$+ (2\tanh^2 x - 1) \begin{pmatrix} -(r_1 + r_2)\mu(r) - \mu(r)(1+\mu(r))r_1 \\ (r_1 - r_2)\mu(r) - \mu(r)(1+\mu(r))r_2 \end{pmatrix}$$

and the stability of M_0 for (25) is equivalent to stability of zero solution for (26).

3.3 New system of unknown

In the case of a nanowire with round section, the Landau-Lifschitz system (24) is invariant by translation in the x-variable and by rotation around the axis of the wire, that is: if m is a solution of (24) then for all $\sigma \in \mathbb{R}$ and $\theta \in \mathbb{R}$, then $(t,x) \mapsto R_\theta(m(t,x-\sigma))$ is also solution for (24). Therefore for all $(\sigma,\theta) \in \mathbb{R}^2$, $x \mapsto R_\theta M_0(x-\sigma)$ is a static solution for (24) and by projection of the mobile frame, we obtain the existence of a two-parameter family of static solutions for (26) given for $\Lambda = (\sigma,\theta)$ by

$$R_\Lambda(x) = \begin{pmatrix} R_\theta M_0(x-\sigma)\cdot M_1(x) \\ \\ R_\theta M_0(x-\sigma)\cdot M_2 \end{pmatrix} = \begin{pmatrix} -\dfrac{\tanh(x-\sigma)}{\cosh x} + \dfrac{\cos\theta \tanh x}{\cosh(x-\sigma)} \\ \\ \dfrac{\sin\theta}{\cosh(x-\sigma)} \end{pmatrix} \tag{27}$$

The existence of this two parameter family of static solutions induces that 0 is a double eigenvalue for the linearized operator $\mathcal{L}$ arising in (26) and the kernel of $\mathcal{L}$ is generated by $\varphi_1 = \begin{pmatrix} 1/\cosh x \\ 0 \end{pmatrix}$ and $\varphi_2 = \begin{pmatrix} 0 \\ 1/\cosh x \end{pmatrix}$.

In order to address the difficulty arising from the null eigenspace, we decompose the solutions of (26) on the form:

$$r(t,x) = R_{\Lambda(t)}(x) + W(t,x), \tag{28}$$

with $\left\langle W_1 | \frac{1}{\cosh x} \right\rangle = \left\langle W_2 | \frac{1}{\cosh x} \right\rangle = 0$.

As it is proved in [6] this is a valid system of coordinates in a neighborhood of zero in $H^2(\mathbb{R})$.

By plugging (28) in (26) we obtain that

$$\partial_\theta R_\Lambda \partial_t \theta + \partial_\sigma R_\Lambda \partial_t \sigma + \partial_t W = \mathcal{L} R_\Lambda + \mathcal{L} W + F_0(x, R_\Lambda, \partial_x R_\Lambda, \partial_{xx} R_\Lambda) + G_0(x, \Lambda, W, \partial_x W, \partial_{xx} W),$$

where $G_0(x,\Lambda,W,\partial_x W,\partial_{xx}W) = F_0(x, R_\Lambda + W, \partial_x R_\Lambda + \partial_x W, \partial_{xx} R_\Lambda + \partial_{xx} W) - F_0(x, R_\Lambda, \partial_x R_\Lambda, \partial_{xx} R_\Lambda)$.
As remarked before, for a fixed Λ, R_Λ is a static solution for (26) so that

$$\mathcal{L} R_\Lambda + F_0(x, R_\Lambda, \partial_x R_\Lambda, \partial_{xx} R_\Lambda) = 0.$$

Therefore we obtain

$$\partial_\sigma R_\Lambda \partial_t \sigma + \partial_\theta R_\Lambda \partial_t \theta + \partial_t W = \mathcal{L} W + G_0(x, \Lambda, W, \partial_x W, \partial_{xx} W). \tag{29}$$

We take the inner product of (29) with φ_1 and φ_2. We remark that $\langle \varphi_i | \partial_t W \rangle = 0$ and that $\langle LW | \varphi_i \rangle = 0$ so that we obtain

$$\mathcal{A}(\Lambda) \frac{d\Lambda}{dt} = v\tilde{\lambda} W + \widetilde{G_0}$$

where

$$\mathcal{A}(\Lambda) = \begin{pmatrix} \langle \partial_\sigma R_\Lambda | \varphi_1 \rangle & \langle \partial_\theta R_\Lambda | \varphi_1 \rangle \\ \\ \langle \partial_\sigma R_\Lambda | \varphi_2 \rangle & \langle \partial_\theta R_\Lambda | \varphi_2 \rangle \end{pmatrix}, \quad \tilde{\lambda} W = \begin{pmatrix} -\langle lW_2 | \frac{1}{\cosh x} \rangle \\ \\ \langle lW_1 | \frac{1}{\cosh x} \rangle \end{pmatrix}, \quad \widetilde{G_0} \begin{pmatrix} \langle G_0 | \varphi_1 \rangle \\ \\ \langle G_0 | \varphi_2 \rangle \end{pmatrix}$$

We remark that the matrix $\mathcal{A}(\Lambda)$ depends continuously on Λ and that

$$\mathcal{A}(0,0) = \begin{pmatrix} 2 & 0 \\ \\ 0 & 2 \end{pmatrix}$$

so there exists $\eta_0 > 0$ and a constant C such that if $|\Lambda| \leq \eta_1$, $\mathcal{A}(\Lambda)$ is invertible with $|\left(\mathcal{A}(\Lambda)\right)^{-1}| \leq C$.

Therefore we obtain the following equation for Λ:

$$\frac{d\Lambda}{dt} = \mathcal{B}(\Lambda, W). \tag{30}$$

where $\mathcal{B}(\Lambda, W) = (\mathcal{A}(\Lambda))^{-1}\left(v\tilde{\lambda}W + \widetilde{G_0}\right)$. We denote $\mathcal{B}_1$ and $\mathcal{B}_2$ the coordinates of $\mathcal{B}$.

By subtraction we obtain that W satisfies

$$\partial_t W = \mathcal{L}W + G_0 - \mathcal{B}_1\partial_\sigma R_\Lambda - \mathcal{B}_2\partial_\theta R_\Lambda. \tag{31}$$

3.4 Estimate of the linear terms

The linear operator $\mathcal{L}$ writes

$$\mathcal{L} = \begin{pmatrix} -L & -L - vl \\ L + vl & -L \end{pmatrix}$$

with $L = -\partial_{xx} + (2\tanh^2 x - 1)$ and $l = \partial_x + \tanh x$.
From Corollary 1, we recall that for $w \in (\frac{1}{\cosh x})^{\perp}$,

$$\|w\|_{L^2}^2 \leq \langle Lw|w\rangle \leq \|Lw\|_{L^2}^2.$$

In order to estimate W we will multiply (26) by $\begin{pmatrix} LW_1 \\ LW_2 \end{pmatrix}$ and it will appear the term $v\left(\langle lW_1|LW_2\rangle - \langle lW_2|LW_1\rangle\right)$.

Let us estimate this term:

Proposition 8. *For W_1 and W_2 in $(\frac{1}{\cosh x})^{\perp}$,*

$$|\langle W_1|LW_2\rangle - \langle LW_1|lW_2\rangle| \leq \frac{1}{2}(\|LW_1\|_{L^2}^2 + \|LW_2\|_{L^2}^2).$$

Proof. Let us denote by $\hat{g}$ the Fourier transform of a function g. We denote by ξ the Fourier variable in $\mathbb{R}^2$. To obtain the proof of the proposition, we will use the fact that $L = l^*l$ and $ll^* = -\partial_{xx} + 1$. Indeed let us denote $d_i = lW_i$ for $i = 1, 2$, then

$$\begin{aligned} \|LW_i\|_{L^2}^2 &= \langle LW_i|LW_i\rangle \\ &= \langle l^*d_i|l^*d_i\rangle \\ &= \langle ll^*d_i|d_i\rangle \\ &= \langle -\partial_{xx}d_i + d_i|d_i\rangle \\ &= \langle(|\xi|^2 + 1)\hat{d}_i|\hat{d}_i\rangle \\ &= \|\sqrt{1 + |\xi|^2}\hat{d}_i\|_{L^2}^2. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \langle lW_1|LW_2\rangle - \langle LW_1|lW_2\rangle &= \langle d_1|l^*d_2\rangle - \langle d_2|l^*d_1\rangle \\ &= 2\langle\partial_x d_1|d_2\rangle. \end{aligned}$$

By young inequality we get

$$\begin{aligned}
|\langle lW_1|LW_2\rangle - \langle LW_1|lW_2\rangle| &\leq 2|\langle \hat{d}_2|i\varepsilon d_1\rangle| \\
&\leq 2\int_{\xi\in\mathbb{R}} |\hat{d}_1||\hat{d}_2||\xi| \\
&\leq \int_{\mathbb{R}} \sqrt{1+|\xi|^2}|\hat{d}_1|\sqrt{1+|\xi|^2}|\hat{d}_2| \\
&\leq \|\sqrt{1+|\xi|^2}|\hat{d}_1|\|_{L^2}\|\sqrt{1+|\xi|^2}|\hat{d}_2|\|_{L^2} \\
&\leq \frac{1}{2}(\|\sqrt{1+|\xi|^2}|\hat{d}_1|\|_{L^2}^2 + \|\sqrt{1+|\xi|^2}|\hat{d}_2|\|_{L^2}^2 \\
&\leq \frac{1}{2}\|LW\|_{L^2}^2.
\end{aligned}$$

□

3.5 Estimate of the non linear terms

We first estimate F_0 arising in Proposition 7. We recall that F_0 writes

$$F_0(x, r, \partial_x r, \partial_{xx} r) = A_0(r)\partial_{xx} r + B_0(x,r)\partial_x r + C_0(r)(\partial_x r, \partial_x r) + D_0(x,r)$$

The properties concerning A_0, B_0, C_0, and D_0 are described in the following proposition:

Proposition 9. *There exists a constant $C > 0$ such that for all r in $B(0, \frac{1}{2})$ and for $x \in \mathbb{R}$ we get:*

- $|A_0(r)| \leq C|r|^2$ *and* $|A_0'(r)| \leq C|r|$,
- $|B_0(x,r)| \leq C|r|$ *and* $|B_0'(r)| \leq C$,
- $|C_0(r)| \leq C|r|$ *and* $|\frac{\partial}{\partial r} C_0(r)| \leq C$,
- $|D_0(x,r)| \leq C|r|^2$, *and* $|\frac{\partial}{\partial r} D_0(x,r)| \leq C|r|$,

Proof. These estimates come from the expression of the terms A_0, B_0, ... □

As in the previous section (for the estimate of G), writing that $G_0(x, \Lambda, W, \partial_x W, \partial_{xx} W) = F_0(x, R_\Lambda + W, \partial_x R_\Lambda + \partial_x W, \partial_{xx} R_\Lambda + \partial_{xx} W) - F_0(x, R_\Lambda, \partial_x R_\Lambda, \partial_{xx} R_\Lambda)$, using the fondamental theorem of Analysis and Proposition 9, we obtain the following estimate: there exist η_0 and a constant C such that if $|\sigma| + |\theta| + \|W\|_{L^\infty} \leq \eta_0$,

$$\|G_0\|_{L^2} \leq C|\sigma|\|W\|_{H^2} + C\|W\|_{H^1}\|W\|_{H^2}. \tag{32}$$

The term $\widetilde{G_0}$ is estimated as we did for $\widetilde{G}$ in the previous section: there exists a constant C such that while $|\sigma| + |\theta| + \|W\|_{L^\infty} \leq \eta_0$,

$$\left|\widetilde{G_0}\right| \leq C\|W\|_{H^1} + C\|W\|_{H^1}^2. \tag{33}$$

In addition, we have

$$|\tilde{\lambda} W| \leq C\|W\|_{H^1}. \tag{34}$$

From (33) and (34), since the matrix $\mathcal{A}(\Lambda)$ is invertible, using the equivalence of norms in Corollary 1, we obtain that there exists η_1 such that while $|\sigma| \leq \eta_1$, $|\theta| \leq \eta_1$ and $\langle LW|W\rangle^{\frac{1}{2}} \leq \eta_1$, we have:

$$|\mathcal{B}(\Lambda, W)| \leq C\langle LW|W\rangle^{\frac{1}{2}}. \tag{35}$$

3.6 End of the proof

Taking the inner product of (31) with $\begin{pmatrix} LW_1 \\ LW_2 \end{pmatrix}$, we obtain:

$$\frac{1}{2}\frac{d}{dt}\langle LW|W\rangle = -\|LW\|_{L^2}^2 + v\left(\langle lW_1|LW_2\rangle - \langle LW_1|lW_2\rangle\right) + \sum_{i=1}^{2}\langle \mathcal{G}_i|LW_i\rangle, \tag{36}$$

where $\mathcal{G} = G_0 - \mathcal{B}_1\partial_\sigma R_\Lambda - \mathcal{B}_2\partial_\theta R_\Lambda$.

From Estimates (32) and (35), using also Proposition 8, we obtain that while $|\sigma| \leq \eta_1$, $|\theta| \leq \eta_1$ and $\langle LW|W\rangle^{\frac{1}{2}} \leq \eta_1$, we have:

$$\frac{1}{2}\frac{d}{dt}\langle LW|W\rangle + \|LW\|_{L^2}^2 \leq \frac{|v|}{2}\|LW\|_{L^2}^2 + C\left(|\Lambda| + \sqrt{\langle LW|W\rangle}\right)\|LW\|_{L^2}^2$$

that is

$$\frac{1}{2}\frac{d}{dt}\langle LW|W\rangle + \|LW\|_{L^2}^2\left(1 - \frac{|v|}{2} - C\left(|\sigma| + |\theta| + \sqrt{\langle LW|W\rangle}\right)\right) \leq 0. \tag{37}$$

So while $|\sigma| \leq \min(\eta_1, \frac{1-\frac{|v|}{2}}{4C})$, $|\theta| \leq \min(\eta_1, \frac{1-\frac{|v|}{2}}{4C})$ and $\sqrt{\langle LW|W\rangle} \leq \min(\eta_1, \frac{1-\frac{|v|}{2}}{4C})$, we have:

$$\frac{1}{2}\frac{d}{dt}\langle LW|W\rangle + \|LW\|_{L^2}^2\frac{1-\frac{|v|}{2}}{4C} \leq 0$$

so that, since $\langle LW|W\rangle \leq \|LW\|_{L^2}^2$ and by comparison lemma,

$$\langle LW|W\rangle(t) \leq \langle LW|W\rangle(0)\exp\left(-\frac{4Ct}{1-\frac{|v|}{2}}\right). \tag{38}$$

Taking into account this exponential decay in Equation (30) using Estimate (35), we conclude the proof of Theorem 1.2 with the same arguments as in the proof of Theorem 1.

4 Proof of Theorem 1.3

The instability result described in Theorem 1.3 can be obtained if we prove that the spectrum of the linear part $\mathcal{L}$ has at least one element λ with strictly positive real part. At first, we recall the following result:

Theorem 4.1. *Let A be an unbounded operator on a Banach space. Let $\lambda \in \mathbb{C}$. We assume that there exists a sequence $(U_n)_n$ such that $\|W_n\| = 1$ and $\|(A - \lambda Id)W_n\|$ tends to 0 when n tends to infinity. Then λ is in the spectrum of A.*

Let us consider a smooth function $\varphi : \mathbb{R} \to [0,1]$ such that $\varphi(x) = 0$ for $x \in [-1,1]$ and $\varphi(x) = 1$ for $x \in [-\frac{1}{2}, \frac{1}{2}]$.

We define the sequence $(u_n)_n$ by:

$$u_n(x) = \begin{cases} 0 & \text{for } x \leq -a_n - \frac{n}{2} - 1 \text{ or } x \geq -a_n + \frac{n}{2} + 1, \\ \varphi\left(x + a_n + \frac{n}{2}\right) & \text{for } -a_n - \frac{n}{2} - 1 \leq x \leq -a_n - \frac{n}{2}, \\ 1 & \text{for } -a_n - \frac{n}{2} \leq x \leq -a_n + \frac{n}{2}, \\ \varphi\left(x + a_n - \frac{n}{2}\right) & \text{for } -a_n + \frac{n}{2} \leq x \leq -a_n + \frac{n}{2} + 1. \end{cases}$$

where $(a_n)_n$ will be fixed below.
Let us consider the sequence $(v_n)_n$ defined by $v_n(x) = u_n(x)e^{i\frac{vx}{2}}$.
Taking $V_n(x) = \begin{pmatrix} v_n(x) \\ iv_n(x) \end{pmatrix}$, we obtain that for all $\lambda \in \mathbb{C}$, we have:

$$(\mathcal{L} - \lambda)V_n = \begin{pmatrix} (-L-\lambda)v_n - i(L+vl)v_n \\ \\ (L+vl)v_n - i(L+\lambda)v_n \end{pmatrix} = \begin{pmatrix} 1 \\ \\ i \end{pmatrix} G_n(x)$$

where $G_n(x) = (-(1+i)L - ivl - \lambda)\, v_n$. On the interval $[-a_n - \dfrac{n}{2}, -a_n + \dfrac{n}{2}]$, $v_n(x) = e^{i\frac{vx}{2}}$ so that

$$G_n(x) = \left(\frac{v^2}{4} - 1 - i\left(\frac{v^2}{4} + 1 + v\right) - \lambda - (1+i)(2\tanh^2 x - 2) - iv(\tanh x + 1) \right) e^{i\frac{vx}{2}}$$

We set

$$\lambda = \frac{v^2}{4} - 1 - i\left(\frac{v^2}{4} + 1 + v\right)$$

so that $G_n(x) = -(1+i)(2\tanh^2 x - 2) - iv(\tanh x + 1)$ on the interval $[-a_n - \dfrac{n}{2}, -a_n + \dfrac{n}{2}]$.

Since $\tanh x$ tends to -1 when x tends to $-\infty$, then for all n, there exists M_n such that for all $x < -M_n$,

$$\left| -(1+i)(2\tanh^2 x - 2) - iv(\tanh x + 1) \right| \leq \frac{1}{n}.$$

We set now $a_n = M_n + \frac{n}{2}$. We have the following estimates:

- for $x \leq -a_n - \dfrac{n}{2} - 1$ or $x \geq -a_n + \dfrac{n}{2} + 1$, $G_n(x) = 0$,
- for $-a_n - \dfrac{n}{2} - 1 \leq x \leq -a_n - \dfrac{n}{2}$ or $-a_n + \dfrac{n}{2} \leq x \leq -a_n + \dfrac{n}{2} + 1$, $|G_n(x)| \leq K$ where K does not depend on n,
- for $-a_n - \dfrac{n}{2} \leq x \leq -a_n + \dfrac{n}{2}$, $|G_n(x)| \leq \frac{1}{n}$.

So, $\|G_n\|_{L^2} \leq (2K^2 + 1)^{\frac{1}{2}}$.

Now we define W_n by $W_n = \dfrac{V_n}{\|V_n\|_{L^2}}$. We have

$$\|(\mathcal{L} - \lambda)W_n\|_{L^2} = \frac{\sqrt{2}}{\|V_n\|_{L^2}} \|G_n\|_{L^2} \leq \sqrt{2}\frac{(2K^2+1)^{\frac{1}{2}}}{\|V_n\|_{L^2}}.$$

Since $\|V_n\|_{L^2}$ tends to zero when n tends to $+\infty$, then $\|(\mathcal{L} - \lambda)W_n\|_{L^2}$ tends to zero when n tends to $+\infty$ whereas $\|W_n\|_{L^2} = 1$. Therefore, λ is in the spectrum of $\mathcal{L}$ by Theorem 4.1. Since the real part of λ equals $\frac{v^2}{4} - 1 > 0$ as $|v| > 2$, we obtain the linear instability of m_v and we conclude the proof of Theorem 1.3.

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