Walker Regime for Walls in Ferromagnetic Nanotubes

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Abstract. Ferromagnetic nanotubes are proposed as an alternative to ferromagnetic nanowires for data-storage applications. In this paper, we consider a two-dimensional model for such devices and we establish the stability of moving walls in the Walker regime when the tube is subject to a small magnetic field.

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1 Introduction

Domain walls formation and propagation in ferromagnetic nanowires are intensively studied. Indeed, their possible applications for data recording (see [19]) or in nano-electronics (see [1]) are very promising. Such devices are modeled by a 1d-Landau-Lifschitz equation, and existence and stability of one-wall profiles are established (see [10, 11, 12, 21] and the references therein).
In [24], the authors propose to use ferromagnetic nanotubes instead of ferromagnetic nanowires or nano strips in order to deal with domain wall motion in the Walker regime, which is stabler and more reliable for applications. In the present work we exhibit a 2d-model for ferromagnetic nanotubes and we study domain wall dynamics in this model for a small applied magnetic field.

Let us recall the 3-dimensional model for a ferromagnetic sample $\mathcal{O} \subset I\!R^3$. We denote by $(u \cdot v)$ the canonical scalar product of u by v in $I\!R^3$ and by $|.|$ the associated norm. The canonical basis of $I\!R^3$ is denoted by (e_1, e_2, e_3) and $\times$ is the usual cross product.

Ferromagnetic materials are characterized by a spontaneous magnetization described by the magnetic moment M defined on $I\!R^+ \times \mathcal{O}$ and satisfying the saturation constraint

$$|M(\mathrm{t}, x)| = M_s \ \ a.e., \tag{1.1}$$

where M_s is constant. The magnetic moment satisfies the Landau-Lifschitz equation

$$\frac{\partial M}{\partial \mathrm{t}} = -\gamma M \times H_e - \frac{\alpha\gamma}{M_s} M \times (M \times H_e), \tag{1.2}$$

in which $\gamma > 0$ is the gyromagnetic ratio, $\alpha > 0$ is the damping coefficient, H_e is the effective field given by:

$$H_e = \frac{A}{\mu_0 M_s^2} \Delta M + H_d(M) + H_{app}. \tag{1.3}$$

Here, $A > 0$ is the exchange coefficient, μ_0 is the permeability of the vacuum, H_{app} is the applied magnetic field, and $H_d(M)$ is the demagnetizing field generated by the magnetization M. In the quasi-stationary model, the operator H_d is given by

$$\begin{cases} \text{div } (H_d(M) + \overline{M}) = 0, \\ \text{curl } H_d(M) = 0, \end{cases} \tag{1.4}$$

where $\overline{M}$ is the extension of M by zero outside $\mathcal{O}$.

The energy associated to a configuration M is given by:

$$\mathcal{E}(M) = \frac{A}{2M_s^2}\int_{\mathcal{O}} |\nabla M|^2 dx + \frac{\mu_0}{2}\int_{I\!R^3} |H_d(M)|^2\, dx - \mu_0 \int_{\mathcal{O}} H_a \cdot M\, dx,$$

and we have $H_e = -\frac{1}{\mu_0}\partial_M \mathcal{E}$.

Existence of weak or strong solutions for (1.2) is addressed in several papers (see [2, 6, 7, 8, 13, 15, 18, 22]).

We focus now on the case of a thin nanotube of axis $I\!Re_1$ with circular section. The nanotube is assimilated to the cylinder $I\!R \times \rho S^1 = \left\{(\mathrm{x},\mathrm{y},\mathrm{z}) \in I\!R^3, \quad \mathrm{y}^2+\mathrm{z}^2 = \rho^2\right\}$. We assume that a magnetic field H_{app} is applied in the direction of the tube axis: $H_{app} = H_a e_1$, $H_a \in I\!R$. We use the two-dimensional model of ferromagnetic thin film obtained in [5] and [14], in which the demagnetizing field reduces to an anisotropic local term forcing M to be tangent to the thin domain. In the case of our nanotube the demagnetizing field is described by the term $-(M\cdot \mathrm{n})\mathrm{n}$, derived from the limit demagnetizing energy $\dfrac{\mu_0}{4}\displaystyle\int_{I\!R\times S^1} |M\cdot \mathrm{n}|^2 d\sigma$, where n is the unit normal vector to the cylinder surface.

In cylindrical coordinates, we write $\mathrm{y} = \rho\cos y$ and $\mathrm{z} = \rho \sin y$, and we obtain the following 2d model:

$$\begin{cases} M : (t,\mathrm{x},y) \to S^2, \quad 2\pi\text{-periodic in the variable } y, \\[2mm] \dfrac{\partial M}{\partial \mathrm{t}} = -\gamma M \times h(M) - \dfrac{\alpha\gamma}{M_s} M \times (M \times h(M)), \\[2mm] h(M) = \dfrac{A}{\mu_0 M_s^2}\dfrac{\partial^2 M}{\partial \mathrm{x}^2} + \dfrac{A}{\mu_0 M_s^2 \rho^2}\dfrac{\partial^2 M}{\partial y^2} - (M\cdot \mathrm{n}(y))\mathrm{n}(y) + H_a e_1, \end{cases} \tag{1.5}$$

where the unit normal vector n is given by $\mathrm{n}(y) = \begin{pmatrix} 0 \\ \cos y \\ \sin y \end{pmatrix}$.

We denote $\mathrm{n}^\perp(y) = \begin{pmatrix} 0 \\ -\sin y \\ \cos y \end{pmatrix}$. By the rescaling $t = \frac{\gamma A \mathrm{t}}{\mu_0 M_s \rho^2}$ and $x = \frac{\mathrm{x}}{\rho}$, we describe M in the frame $(e_1, \mathrm{n}(y), \mathrm{n}^\perp(y))$ writing:

$$M(t,\mathrm{x},y) = M_s\left(\mathbf{m}_1\left(\frac{\gamma A \mathrm{t}}{\mu_0 M_s \rho^2}, \frac{\mathrm{x}}{\rho}, y\right)e_1 + \mathbf{m}_2\left(\frac{\gamma A \mathrm{t}}{\mu_0 M_s \rho^2}, \frac{\mathrm{x}}{\rho}, y\right)\mathrm{n}(y) + \mathbf{m}_3\left(\frac{\gamma A \mathrm{t}}{\mu_0 M_s \rho^2}, \frac{\mathrm{x}}{\rho}, y\right)\mathrm{n}^\perp(y)\right).$$

We obtain that M satisfies (1.5) if and only if $\mathbf{m} = \begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \\ \mathbf{m}_3 \end{pmatrix}$ satisfies

$$\begin{cases} \mathbf{m} : (t,x,y) \to S^2, \quad 2\pi\text{-periodic in the variable } y, \\[2mm] \dfrac{\partial \mathbf{m}}{\partial t} = -\mathbf{m} \times \mathbf{h}(\mathbf{m}) - \alpha \mathbf{m} \times (\mathbf{m} \times \mathbf{h}(\mathbf{m})), \\[2mm] \mathbf{h}(\mathbf{m}) = \partial_{xx}\mathbf{m} + \partial_{yy}\mathbf{m} + 2e_1 \times \partial_y \mathbf{m} + \mathbf{m}_1 e_1 - \kappa \mathbf{m}_3 e_3 + h_a e_1, \end{cases} \tag{1.6}$$

where $\kappa = \dfrac{\mu_0 M_s^2 \rho^2}{A}$ and $h_a = \dfrac{\mu_0 M_s \rho^2}{A} H_a$.

Remark 1.1. *In our model of ferromagnetic thin layer, the demagnetizing field behaves like the planar anisotropy term $-\kappa \mathbf{m}_3 e_3$. The curvature of the tube induces another anisotropic effect since the tube axis $I\!Re_1$ becomes an easy axis of magnetization modeled by the term $+\mathbf{m}_1 e_1$ in the resulting effective field $\mathbf{h}(\mathbf{m})$.*

We deal with domain wall profiles in the Walker regime as in [24]. For a vanishing applied field ($h_a = 0$), we observe the formation of domains in which the magnetization is along the tube axis. One-wall configuration separating a $-e_1$ domain and a $+e_1$ domain is described by the steady state solution $\mathbf{M}_0$ given by

$$\mathbf{M}_0(x) = \begin{pmatrix} \tanh x \\ 1/\cosh x \\ 0 \end{pmatrix}. \tag{1.7}$$

Furthermore, a small applied field in the e_1-direction induces wall motion. This situation is described in our model by the solution:

$$\mathbf{M}^{h_a}(t,x,y) = \mathbf{R}_\theta \left(\mathbf{M}_0(\frac{x-ct}{\delta}) \right), \tag{1.8}$$

where we denote by $\mathbf{R}_\theta$ the rotation matrix:

$$\mathbf{R}_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \tag{1.9}$$

and where c, θ and δ depend on h_a as follows:

$$h_a = \alpha\kappa \sin\theta\cos\theta, \quad \frac{c}{\delta} = -\alpha h_a - \kappa\cos\theta\sin\theta, \quad \frac{1}{\delta^2} = 1 + \kappa\sin^2\theta. \tag{1.10}$$

This solution is only defined for $|h_a| \leq \frac{\alpha\kappa}{2}$, since $h_a = \frac{\alpha\kappa}{2}\sin 2\theta$.

Remark 1.2. *This kind of solution only depending on the x-variable is also observed in 1d-models of nanowires with elliptical sections (see [20] and [21]) and in Walker's 3d-model in [23].*

In this paper we establish that the solution $\mathbf{M}^{h_a}$ is stable in the Lyapunov sense. We also prove that $\mathbf{M}^{h_a}$ is asymptotically stable modulo translations in the x-variable.

We use the following notations:

- $\Omega = I\!R \times]0, 2\pi[$,
- $\mathbf{L}^2_p$ is the space of the measurable functions $u : (x,y) \to I\!R^l$ ($l = 1$, 2 or 3) which are 2π-periodic in y, and such that $u \in L^2(\Omega; I\!R^l)$. We denote by $\langle \mid \rangle$ the associated inner product

$$\langle u \mid v \rangle = \int_\Omega (u(x,y) \cdot v(x,y)) \mathrm{d}x\,\mathrm{d}y,$$

  and by $\|\cdot\|_{\mathbf{L}^2_p}$ the associated norm.
- $\mathbf{H}^k_p$ is the space of the measurable functions $u : (x,y) \to I\!R^l$ ($l = 1$, 2 or 3) which are 2π-periodic in y and such that u belongs to the Sobolev space $H^k(\Omega; I\!R^p)$. The associated norm is denoted by $\|\cdot\|_{\mathbf{H}^k_p}$.

Our main result is the following stability theorem:

Theorem 1. *There exists h_{max}, $0 < h_{max} < \frac{\alpha\kappa}{2}$, such that if $|h_a| \leq h_{max}$, then for all $\varepsilon > 0$, there exists $\eta > 0$ such that if $\mathbf{m}$ satisfies (1.6) with $\|\mathbf{m}(0,\cdot) - \mathbf{M}^{h_a}(0,\cdot)\|_{\mathbf{H}^2_p} \leq \eta$, then:*

- *for all $t > 0$, $\|\mathbf{m}(t,\cdot) - \mathbf{M}^{h_a}(t,\cdot)\|_{\mathbf{H}^2_p} \leq \varepsilon$ (stability),*
- *there exists $\sigma_\infty \in I\!R$ such that $\|\mathbf{m}(t,\cdot) - \mathbf{M}^{h_a}(t, \cdot - \sigma_\infty)\|_{\mathbf{H}^2_p}$ tends to 0 when t tends to $+\infty$ (asymptotic stability modulo translations).*

To our knowledge, this work is the first one dealing with the stability of moving walls structures in dimension strictly greater than 1. In the 3d-model of [4], only static walls are studied. In addition, the model in [4] is not complete since the demagnetizing field is unduly simplified in 3d. Here, the model is more convincing since the 2d-model for the demagnetizing field can be justified by asymptotic arguments (see [5] and [14]).

Roughly speaking, in the proof of Theorem 1, we use the techniques developed in [4] and [10]. We have to address several difficulties, some of them being specific to the nanotube case.

The first one comes from the saturation constraint (1.1), since we must consider only perturbations satisfying this constraint. To overcome this problem we use a moving frame technique as in [10] (see Section 2).

In Section 3, we prove a linear stability result. We prove that the linearization around the studied solution is non negative outside its kernel. This kernel is one-dimensional and relates to the invariance by translation of the Landau-Lifschitz type equation (1.6). The coercivity proof is specific to our 2d case and is quite tricky because of the term $e_1 \times \partial_y \mathbf{m}$ in the effective field in (1.6).

Theorem 1 is established in Section 4. The zero eigenvalue due to the translation invariance is responsible for a drift of the perturbation. As in [16, 17] and the references therein, we split the perturbations of $\mathbf{M}^{ha}$ as a translation of $\mathbf{M}^{ha}$ plus a residual term. The linear estimates of Section 3 and variational estimates yield that this remainder term tends exponentially to zero when t tends to $+\infty$.

For the convenience of the reader, we postpone the technical estimates of the nonlinear terms to Section 5.

2 Moving Frame Technique

The magnetic applied field h_a being fixed, we introduce c, δ and θ given by (1.10). Then we write the solution $\mathbf{m}$ of (1.6) on the form:

$$\mathbf{m}(t,x,y) = \mathbf{R}_\theta \mathbf{v}(t, \frac{x-ct}{\delta}, y),$$

where $\mathbf{v} = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{pmatrix}$ satisfies the saturation constraint $|\mathbf{v}| = 1$, so that $\mathbf{m}$ satisfies (1.6) if and only if $\mathbf{v}$ satisfies

$$\partial_t \mathbf{v} - \frac{c}{\delta}\partial_x \mathbf{v} = -\mathbf{v} \times \mathbf{H}(\mathbf{v}) - \alpha \mathbf{v} \times (\mathbf{v} \times \mathbf{H}(\mathbf{v})), \tag{2.11}$$

where

$$\mathbf{H}(\mathbf{v}) = \frac{1}{\delta^2}\partial_{xx}\mathbf{v} + \partial_{yy}\mathbf{v} + 2e_1 \times \partial_y \mathbf{v} + \mathbf{v}_1 e_1 - \kappa(\sin\theta \mathbf{v}_2 + \cos\theta \mathbf{v}_3)(\sin\theta e_2 + \cos\theta e_3) + h_a e_1.$$

In addition, $\mathbf{M}^{h_a}$ is stable for (1.6) if and only if $\mathbf{M}_0$ is stable for (2.11).
Therefore in order to establish Theorem 1, we aim to prove that if $\|\mathbf{v}(0,\cdot) - \mathbf{M}_0(\cdot)\|_{\mathbf{H}^2_p}$ is small enough, then $\|\mathbf{v}(t,\cdot) - \mathbf{M}_0(\cdot)\|_{\mathbf{H}^2_p}$ remains small for all t and there exists σ_∞ such that $\|\mathbf{v}(t,\cdot) - \mathbf{M}_0(\cdot - \sigma_\infty)\|_{\mathbf{H}^2_p}$ tends to zero when t tends to $+\infty$.

In order to deal with perturbations $\mathbf{v}$ of $\mathbf{M}_0$ satisfying the saturation constraint $|\mathbf{v}| = 1$, we use the mobile frame technique developed in [10]. We introduce $\mathbf{M}_1$ and $\mathbf{M}_2$ defined by

$$\mathbf{M}_1(x) = \begin{pmatrix} -1/\cosh x \\ \tanh x \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{M}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and we write $\mathbf{v}$ on the form

$$\mathbf{v}(t,x,y) = \mathbf{M}_0(x) + r_1(t,x,y)\mathbf{M}_1(x) + r_2(t,x,y)\mathbf{M}_2 + \mu(r(t,x,y))\mathbf{M}_0(x),$$

where the new unknown $r : (t,x,y) \mapsto \begin{pmatrix} r_1(t,x,y) \\ r_2(t,x,y) \end{pmatrix} \in I\!R^2$ is 2π-periodic in the y-variable and where μ is chosen so that the saturation constraint in always satisfied:

$$\mu(\xi) = \sqrt{1 - (\xi_1)^2 - (\xi_2)^2} - 1.$$

Plugging this formulation in (2.11) and taking the scalar product with $\mathbf{M}_1$ and $\mathbf{M}_2$, we obtain that $\mathbf{v}$ satisfies (2.11) if and only if r satisfies:

$$\frac{\partial r}{\partial t} = \begin{pmatrix} -\alpha & -1 \\ 1 & -\alpha \end{pmatrix} \mathcal{M} r - h_a \mathcal{L} r + \mathcal{F}(r), \tag{2.12}$$

where the linear operators $\mathcal{M}$ and $\mathcal{L}$ are defined by

$$\mathcal{M} = \begin{pmatrix} \frac{1}{\delta^2} L - \partial_{yy} & 2\tanh x\, \partial_y \\ \\ -2\tanh x\, \partial_y & \frac{1}{\delta^2} L - \partial_{yy} + \kappa \cos 2\theta \end{pmatrix}, \quad \text{with } L = -\partial_{xx} + (2\tanh^2 - 1), \tag{2.13}$$

and

$$\mathcal{L} r = \left(\alpha + \frac{1}{\alpha}\right) \ell r + 2\tanh x\, r_2 \begin{pmatrix} 1 \\ \\ -\dfrac{1}{\alpha} \end{pmatrix}, \quad \text{with } \ell = \partial_x + \tanh x, \tag{2.14}$$

and where the term $\mathcal{F} : \mathbf{H}^2_p \to \mathbf{L}^2_p$ is the non linear contribution (that is $\partial_r \mathcal{F}(0) = 0$). For the convenience of the reader, the expression of $\mathcal{F}$ is postponed to Section 5.

3 Linear Stability

In this part, we study the stability of the zero solution for the linearization of (2.12):

$$\frac{\partial v}{\partial t} = \begin{pmatrix} -\alpha & -1 \\ 1 & -\alpha \end{pmatrix} \mathcal{M} v - h_a \mathcal{L} v. \tag{3.15}$$

The linear operator L in (2.13) appears in several stability proofs concerning one-dimensional models of nanowires (see [9, 10, 11, 12]). It also appears in [4] for the 3d case. We recall the properties of this operator (see [12] for the proofs):

- L is self-adjoint with domain $H^2(I\!R)$,
- we can factorize L on the form $L = \ell^* \circ \ell$ with $\ell = \partial_x + \tanh x$, so that L is positive. In addition,
$$\int_{I\!R} u \cdot Lu = \|L^{\frac{1}{2}} u\|^2_{L^2(I\!R)} = \|\ell u\|^2_{L^2(I\!R)},$$
- Ker $L = I\!R \dfrac{1}{\cosh x}$, and the essential spectrum of L is $[1, +\infty[$,
- 0 in the only eigenvalue of L,
- on $(\text{Ker } L)^\perp$, $L \geq 1$, so that if $\displaystyle\int_{I\!R} \frac{u(x)}{\cosh x} \mathrm{d}x = 0$, then
$$\|u\|_{L^2(I\!R)} \leq \|\ell u\|_{L^2(I\!R)} \leq \|Lu\|_{L^2(I\!R)}. \tag{3.16}$$

We introduce the following notations:

- $\mathbf{L}_p^{2,\perp} = \left\{ w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathbf{L}_p^2, \quad \text{such that} \ \displaystyle\int_{I\!R\times]0,2\pi[} w_1(t,x,y)\frac{1}{\cosh x}\mathrm{d}x\,\mathrm{d}y = 0 \right\}$,
- $\mathbf{H}_p^{k,\perp} = \mathbf{H}_p^k \cap \mathbf{L}_p^{2,\perp}$.

The properties of $\mathcal{M}$ are summarized in the following:

Proposition 3.1. *For all $\theta \in]-\frac{\pi}{4},\frac{\pi}{4}[$, $\mathcal{M}$ is a self-adjoint positive operator for the $\mathbf{L}_p^2$ inner product with domain $\mathcal{D}(\mathcal{M}) = \mathbf{H}_p^2$, and Ker $\mathcal{M} = I\!R\frac{1}{\cosh x}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.*
In addition, $\mathcal{M}$ is non negative on (Ker $\mathcal{M})^\perp = \mathbf{H}_p^{2,\perp}$, and for all fixed $\theta_{max} \in [0,\frac{\pi}{4}[$, there exist constants $\alpha_1 > 0$, $\alpha_2 > 0$ and $\alpha_3 > 0$ (depending on θ_{max}) such that for all $\theta \in]-\theta_{max},\theta_{max}[$,

- *for $k \in \{1,2\}$, for all $w \in \mathbf{H}_p^{k+1,\perp}$, $\alpha_1\|\mathcal{M}^{\frac{k}{2}}w\|_{\mathbf{L}_p^2} \le \|\mathcal{M}^{\frac{k+1}{2}}w\|_{\mathbf{L}_p^2}$,*
- *for $k \in \{1,2,3\}$, for all $w \in \mathbf{H}_p^{k,\perp}$, $\quad \alpha_2\|w\|_{\mathbf{H}_p^k} \le \|\mathcal{M}^{\frac{k}{2}}w\|_{\mathbf{L}_p^2} \le \alpha_3\|w\|_{\mathbf{H}_p^k}$.*

As a corollary of Proposition 3.1, we obtain the following Theorem:

Theorem 2. *There exists h_{max}^l with $0 < h_{max}^l < \dfrac{\alpha\kappa}{2}$, such that for all h_a with $|h_a| \le h_{max}^l$, the zero solution is stable for Equation (3.15). More precisely, for all $\varepsilon > 0$, there exists $\eta > 0$ such that for all $v_0 \in \mathbf{H}_p^1$, if $\|v_0\|_{\mathbf{H}_p^1} \le \eta$, then the solution v of (3.15) with initial data v_0 satisfies*

$$\forall\, t > 0, \quad \|v(t,\cdot)\|_{\mathbf{H}_p^1} \le \varepsilon.$$

In addition, when t tends to $+\infty$, $v(t,\cdot)$ tends in $\mathbf{H}_p^1$ to a limit of the form $\dfrac{\sigma_\infty}{\cosh x}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, where $\sigma_\infty \in I\!R$.

Remark 3.1. *We prove the linear stability in $\mathbf{H}_p^1$. We could also prove the same result in $\mathbf{H}_p^2$. For the nonlinear stability, we need $\mathbf{H}_p^2$ estimates to control the nonlinear terms. Theorem 2 is proved in Section 3.2*

3.1 Proof of Proposition 3.1

We first establish the following Lemma:

Lemma 3.1. *There exists $c_1 > 0$ and $c_2 > 0$ such that for all $w \in \mathbf{H}_p^{1,\perp}$,*

$$c_1\|w\|_{\mathbf{H}_p^1} \le \left(\|\ell w_1\|_{\mathbf{L}_p^2}^2 + \|\partial_y w_1\|_{\mathbf{L}_p^2}^2 + \|\ell w_2\|_{\mathbf{L}_p^2}^2 + \|\partial_y w_2\|_{\mathbf{L}_p^2}^2 + \|w_2\|_{\mathbf{L}_p^2}^2\right)^{\frac{1}{2}} \le c_2\|w\|_{\mathbf{H}_p^1}.$$

Proof. We recall that $\ell = \partial_x + \tanh x$, so that the existence of c_2 is straightforward.
We have

$$\|\partial_x w_2\|_{\mathbf{L}_p^2} \le \|\ell w_2 - \tanh x\, w_2\|_{\mathbf{L}_p^2} \le \|\ell w_2\|_{\mathbf{L}_p^2} + \|\tanh x\, w_2\|_{\mathbf{L}_p^2},$$

thus there exists K such that

$$\|w_2\|_{\mathbf{H}_p^1} \le K\left(\|\ell w_2\|_{\mathbf{L}_p^2} + \|\partial_y w_2\|_{\mathbf{L}_p^2} + \|w_2\|_{\mathbf{L}_p^2}\right). \tag{3.17}$$

Concerning w_1, we first recall that for all $u \in H^1(I\!R)$, we have

$$\int_{I\!R} u(x)\frac{1}{\cosh x}\mathrm{d}x = 0 \quad \Longrightarrow \quad \|u\|_{L^2(I\!R)} \le \|\ell u\|_{L^2(I\!R)} \quad \text{(see (3.16))}.$$

We define τ by

$$\tau(y) = \int_{x\in I\!R} \frac{1}{2\cosh x} w_1(x,y)\mathrm{d}x.$$

We split w_1 as:

$$w_1(x,y) = W_1(x,y) + \frac{\tau(y)}{\cosh x} \qquad \text{with } \forall\, y \in I\!R, \quad \int_{x\in I\!R} W_1(x,y)\frac{1}{\cosh x}\mathrm{d}x = 0. \tag{3.18}$$

For a fixed y, $x \mapsto W_1(x,y)$ is in $(\frac{1}{\cosh x})^\perp$ (for the $L^2(I\!R)$-inner product), so

$$\|W_1(\cdot,y)\|_{L^2(I\!R)} \le \|\ell W_1(\cdot,y)\|_{L^2(I\!R)},$$

and by integration in the variable $y \in [0,2\pi]$, using that $\ell\frac{1}{\cosh x} = 0$, we obtain that

$$\|W_1\|_{\mathbf{L}^2_p} \le \|\ell w_1\|_{\mathbf{L}^2_p}.$$

By the orthogonality condition in (3.18) we remark that $\left\langle W_1 \mid \frac{1}{\cosh x}\right\rangle = 0$ so

$$\|w_1\|^2_{\mathbf{L}^2_p} = \|W_1\|^2_{\mathbf{L}^2_p} + \|\frac{\tau(y)}{\cosh x}\|^2_{\mathbf{L}^2_p}.$$

We have:

$$\|\frac{\tau(y)}{\cosh x}\|^2_{\mathbf{L}^2_p} = \int_{x\in I\!R}\int_{y\in]0,2\pi[} |\tau(y)|^2 \frac{1}{\cosh^2 x}\,\mathrm{d}x\,\mathrm{d}y = 2\|\tau\|^2_{L^2([0,2\pi])}.$$

Since $w \in \mathbf{H}^{1,\perp}_p$, we have $\left\langle w_1 \mid \frac{1}{\cosh x}\right\rangle = 0$ thus

$$\int_{]0,2\pi[} \tau(y)\mathrm{d}y = 0.$$

Hence by Poincaré-Wirtinger inequality, we have

$$\|\tau\|_{L^2(]0,2\pi[)} \le \|\partial_y \tau\|_{L^2(]0,2\pi[)}.$$

Therefore,

$$\begin{aligned}
\|w_1\|^2_{\mathbf{L}^2_p} \le\ & \|\ell w_1\|^2_{\mathbf{L}^2_p} + 2\int_{]0,2\pi[} |\partial_y\tau|^2 \mathrm{d}y,\\
\le\ & \|\ell w_1\|^2_{\mathbf{L}^2_p} + 2\int_{]0,2\pi[} \left|\int_{I\!R} \frac{1}{2\cosh x}\partial_y w_1(x,y)\mathrm{d}x\right|^2 \mathrm{d}y,\\
\le\ & \|\ell w_1\|^2_{\mathbf{L}^2_p} + 2\int_{]0,2\pi[} \left(\int_{I\!R}\frac{1}{4\cosh^2 x}\mathrm{d}x\right)\left(\int_{I\!R}|\partial_y w_1(x,y)|^2\mathrm{d}x\right)\mathrm{d}y\\
& \text{by Cauchy-Schwarz inequality,}
\end{aligned}$$

thus

$$\|w_1\|^2_{\mathbf{L}^2_p} \le \|\ell w_1\|^2_{\mathbf{L}^2_p} + \|\partial_y w_1\|^2_{\mathbf{L}^2_p}. \tag{3.19}$$

Therefore

$$\begin{aligned}
\|w_1\|_{\mathbf{H}^1_p} =\ & \|w_1\|_{\mathbf{L}^2_p} + \|\partial_x w_1\|_{\mathbf{L}^2_p} + \|\partial_y w_1\|_{\mathbf{L}^2_p},\\
\le\ & \|w_1\|_{\mathbf{L}^2_p} + \|\ell w_1\|_{\mathbf{L}^2_p} + \|\tanh x\; w_1\|_{\mathbf{L}^2_p} + \|\partial_y w_1\|_{\mathbf{L}^2_p}, \qquad (3.20)\\
\le\ & 3\left(\|\ell w_1\|_{\mathbf{L}^2_p} + \|\partial_y w_1\|_{\mathbf{L}^2_p}\right) \qquad \text{using (3.19)}.
\end{aligned}$$

Coupling (3.17) and (3.20) we obtain the existence of c_1. This concludes the proof of Lemma 3.1.

□

In order to establish that $\mathcal{M}$ is non negative, we prove the following:

Lemma 3.2. *There exists $c > 0$ such that for all $\theta \in]-\pi/4, \pi/4[$, for all $w \in \mathbf{H}_p^{2,\perp}$, we have*

$$\langle \mathcal{M}w \mid w\rangle \geq c\cos 2\theta \|w\|_{\mathbf{H}_p^1}^2.$$

Proof. We recall that we denote $\Omega = I\!R \times [0, 2\pi]$. We have

$$\begin{aligned}\langle \mathcal{M}w \mid w\rangle = \; & \frac{1}{\delta^2}\langle Lw_1 \mid w_1\rangle + \|\partial_y w_1\|_{\mathbf{L}_p^2}^2 + \frac{1}{\delta^2}\langle Lw_2 \mid w_2\rangle + \|\partial_y w_2\|_{\mathbf{L}_p^2} \\ & +\kappa\cos 2\theta\|w_2\|_{\mathbf{L}_p^2}^2 - 2\int_\Omega \tanh x\, w_2 \partial_y w_1 + 2\int_\Omega \tanh x\, w_1\partial_y w_2.\end{aligned} \tag{3.21}$$

We estimate the last two integrals. Let ν, $0 < \nu \leq 1$. We have:

$$\begin{aligned}-2\int_\Omega \tanh x\, w_2\partial_y w_1 \; +2\int_\Omega \tanh x\, w_1\partial_y w_2 = & -4\nu\int_\Omega \tanh x\, w_2\partial_y w_1 \\ & -2(1-\nu)\int_\Omega \tanh x\, w_2\partial_y w_1 + 2(1-\nu)\int_\Omega \tanh x\, w_1 \partial_y w_2.\end{aligned}$$

Since $\int_\Omega \partial_x w_2\partial_y w_1 - \int_\Omega \partial_x w_1 \partial_y w_2 = 0$, we have

$$-\int_\Omega \tanh x\, w_2\partial_y w_1 + \int_\Omega \tanh x\; w_1\partial_y w_2 = -\int_\Omega \ell w_2 \partial_y w_1 + \int_\Omega \ell w_1\partial_y w_2$$

(we recall that $\ell w_i = \partial_x w_i + \tanh x\, w_i$). Therefore

$$\begin{aligned}-2\int_\Omega \tanh x\, w_2\partial_y w_1 + 2\int_\Omega \tanh x\, w_1 \partial_y w_2 = \; & -4\nu\int_\Omega \tanh x\, w_2\partial_y w_1 \\ & -2(1-\nu)\int_\Omega (\ell w_2\partial_y w_1 - \ell w_1\partial_y w_2).\end{aligned}$$

Hence

$$\begin{aligned}\left|-2\int_\Omega \tanh x\, w_2\partial_y w_1 \; +2\int_\Omega \tanh x\, w_1\partial_y w_2\right| & \leq 4\nu\|\partial_y w_1\|_{\mathbf{L}_p^2}\|w_2\|_{\mathbf{L}_p^2} \\ & +2(1-\nu)\|\ell w_1\|_{\mathbf{L}_p^2}\|\partial_y w_2\|_{\mathbf{L}_p^2} + 2(1-\nu)\|\ell w_2\|_{\mathbf{L}_p^2}\|\partial_y w_1\|_{\mathbf{L}_p^2}, \\ \leq \; & \frac{\nu}{2}\|\partial_y w_1\|_{\mathbf{L}_p^2}^2 + 8\nu\|w_2\|_{\mathbf{L}_p^2}^2 + (1-\nu)\|\ell w_1\|_{\mathbf{L}_p^2}^2 + (1-\nu)\|\partial_y w_2\|_{\mathbf{L}_p^2}^2 \\ & +(1-\nu)\|\ell w_2\|_{\mathbf{L}_p^2}^2 + (1-\nu)\|\partial_y w_1\|_{\mathbf{L}_p^2}^2, \\ \leq \; & (1-\nu)\|\ell w_1\|_{\mathbf{L}_p^2}^2 + (1-\frac{\nu}{2})\|\partial_y w_1\|_{\mathbf{L}_p^2}^2 + (1-\nu)\|\partial_y w_2\|_{\mathbf{L}_p^2}^2 \\ & +(1-\nu)\|\ell w_2\|_{\mathbf{L}_p^2}^2 + 8\nu\|w_2\|_{\mathbf{L}_p^2}^2.\end{aligned}$$

So, using this estimate in (3.21), since $\langle Lw_i \mid w_i\rangle = \|\ell w_i\|_{\mathbf{L}_p^2}^2$, since $\frac{1}{\delta^2} - 1 = \kappa\sin^2\theta$, we have

$$\begin{aligned}\langle \mathcal{M}w \mid w\rangle \geq \; & (\kappa\sin^2\theta + \nu)\|\ell w_1\|_{\mathbf{L}_p^2}^2 + \frac{\nu}{2}\|\partial_y w_1\|_{\mathbf{L}_p^2}^2 + (\kappa\sin^2\theta+\nu)\|\ell w_2\|_{\mathbf{L}_p^2}^2 \\ & +\nu\|\partial_y w_2\|_{\mathbf{L}_p^2}^2 + (\kappa\cos 2\theta - 8\nu)\|w_2\|_{\mathbf{L}_p^2}^2.\end{aligned}$$

We take $\nu = \nu(\theta) := \inf \left\{1, \dfrac{\kappa \cos 2\theta}{16}\right\}$ and we obtain that

$$\langle \mathcal{M}w \mid w\rangle \geq \frac{\nu(\theta)}{2}\left(\|\ell w_1\|^2_{\mathbf{L}^2_p} + \|\partial_y w_1\|^2_{\mathbf{L}^2_p} + \|\ell w_2\|^2_{\mathbf{L}^2_p} + \|\partial_y w_2\|^2_{\mathbf{L}^2_p} + \|w_2\|^2_{\mathbf{L}^2_p}\right).$$

Using Lemma 3.1 we obtain:

$$\langle \mathcal{M}w \mid w\rangle \geq \frac{(c_1)^2}{2}\nu(\theta)\|w\|^2_{\mathbf{H}^1_p}.$$

We remark now that there exists $c > 0$ such that for all $\theta \in]-\pi/4, \pi/4[$,

$$\frac{(c_1)^2}{2}\nu(\theta) \geq c\cos 2\theta.$$

This concludes the proof of Lemma 3.2.

□

Proof of Proposition 3.1. For all $\theta \in]-\theta_{max}, \theta_{max}[$, $\cos 2\theta \geq \cos 2\theta_{max}$. We set

$$\alpha_1 = (c\cos 2\theta_{max})^{\frac{1}{2}}$$

(see Lemma 3.2) and we have, by density of $\mathbf{H}^{2,\perp}_p$ in $\mathbf{H}^{1,\perp}_p$, that

$$\forall\, w \in \mathbf{H}^{1,\perp}, \quad \alpha_1\|w\|_{\mathbf{H}^1_p} \leq \|\mathcal{M}^{\frac{1}{2}}w\|_{\mathbf{L}^2_p}, \tag{3.22}$$

thus

$$\forall\, w \in \mathbf{H}^{1,\perp}, \quad \alpha_1\|w\|_{\mathbf{L}^2_p} \leq \|\mathcal{M}^{\frac{1}{2}}w\|_{\mathbf{L}^2_p}. \tag{3.23}$$

In addition,

$$\alpha_1\|\mathcal{M}^{\frac{1}{2}}w\|^2_{\mathbf{L}^2_p} = \alpha_1\langle w \mid \mathcal{M}w\rangle \leq \alpha_3\|w\|_{\mathbf{L}^2_p}\|\mathcal{M}w\|_{\mathbf{L}^2_p} \leq \|\mathcal{M}^{\frac{1}{2}}w\|_{\mathbf{L}^2_p}\|\mathcal{M}w\|_{\mathbf{L}^2_p}.$$

We obtain then that

$$\alpha_1\|\mathcal{M}^{\frac{1}{2}}w\|_{\mathbf{L}^2_p} \leq \|\mathcal{M}w\|_{\mathbf{L}^2_p}.$$

If $w \in \mathbf{H}^{3,\perp}_p$, then $\mathcal{M}w \in \mathbf{H}^{1,\perp}_p$. Therefore, applying (3.23) replacing w by $\mathcal{M}w$, we obtain that for $w \in \mathbf{H}^{3,\perp}_p$,

$$\alpha_1\|\mathcal{M}w\|_{\mathbf{L}^2_p} \leq \|\mathcal{M}^{\frac{1}{2}}w\|_{\mathbf{L}^2_p}. \tag{3.24}$$

The existence of α_3 is straightforward, since $\mathcal{M}$ is an order-two operator.

Concerning α_2, we first remark that from Proposition 3.2 we have:

$$\alpha_1\|w\|_{\mathbf{H}^1_p} \leq \|\mathcal{M}^{\frac{1}{2}}w\|_{\mathbf{L}^2_p}.$$

In addition, writing $\Delta_\delta = \frac{1}{\delta^2}\partial_{xx} + \partial_{yy}$, we have

$$\mathcal{M}w = -\Delta_\delta w + \mathcal{A}(w),$$

where $\mathcal{A}(w) = 2\tanh x \begin{pmatrix} \partial_y w_2 \\ -\partial_y w_1 \end{pmatrix} + \frac{1}{\delta^2}(1 - 2\tanh^2 x)w + \kappa\cos 2\theta \begin{pmatrix} 0 \\ w_2 \end{pmatrix}$.

So

$$\begin{aligned}
\|\Delta_\delta w\|_{\mathbf{L}^2_p} &\leq \|\mathcal{M}w\|_{\mathbf{L}^2_p} + \|\mathcal{A}w\|_{\mathbf{L}^2_p}, \\
&\leq \|\mathcal{M}w\|_{\mathbf{L}^2_p} + c\|w\|_{\mathbf{H}^1_p} \text{ since } \mathcal{A} \text{ is an order-one operator}, \\
&\leq \|\mathcal{M}w\|_{\mathbf{L}^2_p} + \frac{c}{\alpha_1}\|\mathcal{M}^{\frac{1}{2}}w\|_{\mathbf{L}^2_p} \text{ with (3.22)}, \\
&\leq (1 + \frac{c}{(\alpha_1)^2})\|\mathcal{M}w\|_{\mathbf{L}^2_p}.
\end{aligned}$$

Since $\|\Delta_\delta w\|^2_{\mathbf{L}^2_p} = \dfrac{1}{\delta^4}\|\partial_{xx}w\|^2_{\mathbf{L}^2_p} + 2\dfrac{1}{\delta^2}\|\partial_{xy}w\|^2_{\mathbf{L}^2_p} + \|\partial_{yy}w\|^2_{\mathbf{L}^2_p}$ with $1 \leq \dfrac{1}{\delta^2} \leq 2$, we obtain that there exists a constant k independent of θ and w such that

$$\|\partial_{xx}w\|_{\mathbf{L}^2_p} + \|\partial_{xy}w\|_{\mathbf{L}^2_p} + \|\partial_{yy}w\|_{\mathbf{L}^2_p} \leq k\|\mathcal{M}w\|_{\mathbf{L}^2_p}.$$

Since we already know that $\|w\|_{\mathbf{H}^1_p} \leq \dfrac{1}{\alpha_1}\|\mathcal{M}^{\frac{1}{2}}w\|_{\mathbf{L}^2_p} \leq \dfrac{1}{(\alpha_1)^2}\|\mathcal{M}w\|_{\mathbf{L}^2_p}$, we obtain that there exists a constant a_2 such that:

$$\forall\,\theta \in [-\theta_{max}, \theta_{max}],\ \forall\, w \in \mathbf{H}^{2,\perp}_p, \quad a_2\|w\|_{\mathbf{H}^2_p} \leq \|\mathcal{M}w\|_{\mathbf{L}^2_p}.$$

Concerning the H^3 estimate, we remark that

$$\begin{aligned}
\|\mathcal{M}^{\frac{3}{2}}w\|^2_{\mathbf{L}^2_p} &= \left\langle \mathcal{M}^2 w \mid \mathcal{M}w\right\rangle, \\
&= \left\langle \Delta_\delta^2 w \mid \Delta_\delta w\right\rangle + \left\langle \Delta_\delta w \mid \Delta_\delta \mathcal{A}w + \mathcal{A}\Delta_\delta w + \mathcal{A}^2 w\right\rangle.
\end{aligned}$$

So

$$\begin{aligned}
\frac{1}{\delta^2}\|\partial_x\Delta_\delta w\|^2_{\mathbf{L}^2_p} + \|\partial_y\Delta_\delta w\|^2_{\mathbf{L}^2_p} &\leq \|\mathcal{M}^{\frac{3}{2}}w\|^2_{\mathbf{L}^2_p} + |\left\langle \Delta_\delta w \mid \Delta_\delta \mathcal{A}w + \mathcal{A}\Delta_\delta w + \mathcal{A}^2 w\right\rangle|, \\
&\leq \|\mathcal{M}^{\frac{3}{2}}w\|^2_{\mathbf{L}^2_p} + \|\Delta_\delta w\|_{\mathbf{L}^2_p}\|\Delta_\delta \mathcal{A}w + \mathcal{A}\Delta_\delta w + \mathcal{A}^2 w\|_{\mathbf{L}^2_p}, \\
&\leq \|\mathcal{M}^{\frac{3}{2}}w\|^2_{\mathbf{L}^2_p} + c\|w\|_{\mathbf{H}^2_p}\|w\|_{\mathbf{H}^3_p}.
\end{aligned}$$

We remark that there exists $k_1 > 0$ such that

$$\|w\|_{\mathbf{H}^3_p} \leq k_1\left(\frac{1}{\delta^2}\|\partial_x\Delta_\delta w\|^2_{\mathbf{L}^2_p} + \|\partial_y\Delta_\delta w\|^2_{\mathbf{L}^2_p}\right)^{\frac{1}{2}} + \|w\|_{\mathbf{H}^2_p}, \tag{3.25}$$

so

$$\begin{aligned}
\frac{1}{\delta^2}\|\partial_x\Delta_\delta w\|^2_{\mathbf{L}^2_p} + \|\partial_y\Delta_\delta w\|^2_{\mathbf{L}^2_p} \leq\ & \|\mathcal{M}^{\frac{3}{2}}w\|^2_{\mathbf{L}^2_p} + c\,k_1\|w\|_{\mathbf{H}^2_p}\left(\frac{1}{\delta^2}\|\partial_x\Delta_\delta w\|^2_{\mathbf{L}^2_p} + \|\partial_y\Delta_\delta w\|^2_{\mathbf{L}^2_p}\right)^{\frac{1}{2}} \\
& + c\|w\|^2_{\mathbf{H}^2_p}, \\
\leq\ & (c + \frac{c^2(k_1)^2}{2})\|w\|^2_{\mathbf{H}^2_p} + \|\mathcal{M}^{\frac{3}{2}}w\|^2_{\mathbf{L}^2_p} \\
& + \frac{1}{2}\left(\frac{1}{\delta^2}\|\partial_x\Delta_\delta w\|^2_{\mathbf{L}^2_p} + \|\partial_y\Delta_\delta w\|^2_{\mathbf{L}^2_p}\right) \text{ by Young inequality.}
\end{aligned}$$

Therefore we obtain that

$$\frac{1}{\delta^2}\|\partial_x\Delta_\delta w\|^2_{\mathbf{L}^2_p} + \|\partial_y\Delta_\delta w\|^2_{\mathbf{L}^2_p} \leq 2\|\mathcal{M}^{\frac{3}{2}}w\|^2_{\mathbf{L}^2_p} + 2(c + \frac{c^2(k_1)^2}{2})\|w\|^2_{\mathbf{H}^2_p}.$$

Using (3.25), (3.24) and the previous H^2-estimates, we obtain that there exists $a_3 > 0$ such that

$$a_3\|w\|_{\mathbf{H}^3_p} \leq \|\mathcal{M}^{\frac{3}{2}}w\|_{\mathbf{L}^2_p}.$$

Taking $\alpha_2 = \min(\alpha_1, a_2, a_3)$, we conclude the proof of Proposition 3.1.

□

3.2 Proof of Theorem 2

We fix h_a^{max}, with $0 < h_a^{max} < \frac{\alpha\kappa}{2}$, and $\theta_{max} = \frac{1}{2}\arcsin(\frac{2h_a^{max}}{\alpha\kappa})$. We introduce the constants α_1, α_2, and α_3 given by Proposition 3.1.

We consider v a solution of (3.15). We define $\sigma(t)$ by

$$\sigma(t) = \frac{1}{4\pi}\int_\Omega \frac{v(t,x,y)}{\cosh x} dx\, dy,$$

so that we can split v as

$$v(t,x,y) = w(t,x,y) + \frac{\sigma(t)}{\cosh x}\begin{pmatrix}1\\0\end{pmatrix}, \text{ where } w(t,\cdot) \in H_p^{1,\perp}. \tag{3.26}$$

We plug (3.26) in (3.15). Since $\mathcal{M}\begin{pmatrix}\frac{1}{\cosh x}\\0\end{pmatrix} = \ell\begin{pmatrix}\frac{1}{\cosh x}\\0\end{pmatrix} = 0$, we have:

$$\frac{\partial w}{\partial t} + \frac{1}{\cosh x}\frac{d\sigma}{dt}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-\alpha & -1\\ 1 & -\alpha\end{pmatrix}\mathcal{M}w - h_a\mathcal{L}w. \tag{3.27}$$

We take the L^2-inner product of (3.27) with $\frac{1}{4\pi\cosh x}\begin{pmatrix}1\\0\end{pmatrix}$. We remark that

- $\left\langle w_1 \mid \frac{1}{\cosh x}\right\rangle = 0$ for all t, so $\left\langle \partial_t w_1 \mid \frac{1}{\cosh x}\right\rangle = 0$,
- $\left\langle \partial_{yy} w_i \mid \frac{1}{\cosh x}\right\rangle = \left\langle \tanh x \partial_y w_i \mid \frac{1}{\cosh x}\right\rangle = 0$, by integration of parts in y and 2π-periodicity,
- L is self-adjoint so that $\left\langle Lw_i \mid \frac{1}{\cosh x}\right\rangle = \left\langle w_i \mid L(\frac{1}{\cosh x})\right\rangle = 0$,
- $\left\langle \frac{1}{\cosh x} \mid \frac{1}{4\pi\cosh x}\right\rangle = 1$.

Therefore we obtain:

$$\frac{d\sigma}{dt} = \mathcal{K}w, \tag{3.28}$$

with $\mathcal{K}w = \left\langle \frac{1}{4\pi\cosh x} \mid -\kappa\cos 2\theta\, w_2 - h_a(\alpha + \frac{1}{\alpha})\ell w_1 + 2h_a \tanh x\, w_2\right\rangle$. By subtraction, we obtain that

$$\frac{\partial w}{\partial t} = \begin{pmatrix}-\alpha & -1\\ 1 & -\alpha\end{pmatrix}\mathcal{M}w - h_a\mathcal{L}w - \frac{1}{\cosh x}\begin{pmatrix}1\\0\end{pmatrix}\mathcal{K}w. \tag{3.29}$$

We take the inner product of (3.29) with $\mathcal{M}w$. Since L is self adjoint, since $L(\frac{1}{\cosh x}) = 0$, and by integration by parts in the y variable, we remark that

$$\left\langle \frac{1}{\cosh x} \mid (\mathcal{M}w)_1\right\rangle = \left\langle \frac{1}{\cosh x} \mid \frac{1}{\delta^2}Lw_1 - \partial_{yy}w_1 - 2\tanh x\,\partial_y w_2\right\rangle = 0.$$

Therefore, we obtain that

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{M}^{\frac{1}{2}}w\|_{\mathbf{L}_p^2}^2 + \alpha\|\mathcal{M}w\|_{\mathbf{L}_p^2}^2 = -h_a\left\langle \mathcal{L}w \mid \mathcal{M}w\right\rangle.$$

Since $\mathcal{L}$ is an order one operator, there exists K such that:

$$\|\mathcal{L}w\|_{\mathbf{L}_p^2} \le K\|w\|_{\mathbf{H}_p^1} \le K\|w\|_{\mathbf{H}_p^2}. \tag{3.30}$$

The equivalence of norms in Proposition 3.1 yields:

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{M}^{\frac{1}{2}}w\|^2_{\mathbf{L}^2_p} + \alpha\|\mathcal{M}w\|^2_{\mathbf{L}^2_p} \leq |h_a|\frac{K}{\alpha_2}\|\mathcal{M}w\|^2_{\mathbf{L}^2_p}.$$

We set $h^l_{max} = \inf\left\{h_a^{max}, \frac{\alpha\alpha_2}{2K}\right\}$, and we get that if $|h_a| \leq h^l_{max}$, then for all t,

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{M}^{\frac{1}{2}}w\|^2_{\mathbf{L}^2_p} + \frac{\alpha}{2}\|\mathcal{M}w\|^2_{\mathbf{L}^2_p} \leq 0.$$

Using again Proposition 3.1, we obtain that

$$\frac{d}{dt}\|\mathcal{M}^{\frac{1}{2}}w\|^2_{\mathbf{L}^2_p} + \alpha(\alpha_1)^2\|\mathcal{M}^{\frac{1}{2}}w\|^2_{\mathbf{L}^2_p} \leq 0,$$

so

$$\forall\ t \geq 0, \quad \|\mathcal{M}^{\frac{1}{2}}w\|_{\mathbf{L}^2_p} \leq \|\mathcal{M}^{\frac{1}{2}}w(0)\|_{\mathbf{L}^2_p} e^{-\frac{\alpha(\alpha_1)^2}{2}t}.$$

This implies that:

$$\forall\ t \geq 0, \quad \|w(t)\|_{\mathbf{H}^1_p} \leq \frac{\alpha_3}{\alpha_2}\|w(0)\|_{\mathbf{H}^1_p} e^{-\frac{\alpha(\alpha_1)^2}{2}t}.$$

In addition, using that $|\mathcal{K}w(t)| \leq K\|w(t)\|_{\mathbf{H}^1_p}$ and Equation (3.28), we obtain that $\frac{d\sigma}{dt}$ is integrable on $I\!R^+$, so $\sigma(t)$ tends to a limit σ_∞ when t tends to $+\infty$. This concludes the proof of Theorem 2.

4 Proof of the nonlinear stability

4.1 New unknowns

We remark that our model (2.11) is invariant by translation in the x-variable so that $x \mapsto \mathbf{M}_0(x-\sigma)$ is a static solution for (2.11) for all $\sigma \in I\!R$. By projection on the mobile frame $(\mathbf{M}_1, \mathbf{M}_2)$, this induces the existence of a one-parameter family of static solutions for (2.12) given by

$$R(\sigma)(x) = \begin{pmatrix} \rho(\sigma)(x) \\ 0 \end{pmatrix}, \tag{4.31}$$

where $\rho(s)(x) = (M_0(x-\sigma)\cdot M_1(x)) = -\frac{\tanh(x-s)}{\cosh x} + \frac{\tanh x}{\cosh(x-s)}$.

The existence of this one-parameter family of solutions induces that 0 is in the spectrum of the operator arising in the linearization of (2.12), as observed in Section 3 (see also [4, 10, 11]).

Remark 4.1. *In [9, 10, 11, 12], in the case of wires with circular cross section, the model is also invariant by rotation around the wire axis, so that 0 is an eigenvalue of multiplicity two of the linearization*

In order to take into account this null eigenvalue, we rewrite r in the following new system of coordinates:

$$r(t,x,y) = R(\sigma(t))(x) + w(t,x,y), \tag{4.32}$$

where for all t, $w(t,\cdot) \in \mathcal{H}^{2,\perp}_p$, *i.e.* its first component w_1 satisfies:

$$\forall\, t \geq 0, \quad \int_\Omega w_1(t,x,y)\frac{1}{\cosh x}\mathrm{d}x\,\mathrm{d}y = 0. \tag{4.33}$$

The validity of this system of coordinates is claimed in the following:

Proposition 4.1. *There exists $\xi_0 > 0$ such that for all $r \in \mathbf{L}^2_p$ with $\|r\|_{\mathbf{L}^2_p} \leq \xi_0$, then there exists one and only one couple $(\sigma, w) \in I\!R \times \mathbf{L}^{2,\perp}_p$ such that*

$$r(x,y) = R(\sigma)(x) + w(x,y).$$

In addition, if $r \in \mathbf{H}^k_p$ then $w \in \mathbf{H}^{k,\perp}_p$.

Proof. Proceeding as in [4] we define $\psi : I\!R \to I\!R$ by

$$\psi(s) = \int_{I\!R\times]0,2\pi[} \rho(s)(x)\frac{1}{\cosh x}\mathrm{d}x = 2\pi \int_{x\in I\!R} \left(-\frac{\tanh(x-s)}{\cosh^2 x} + \frac{\tanh x}{\cosh(x-s)\cosh x}\right)\mathrm{d}x.$$

We remark that if r admits a decomposition on the form $r(x,y) = R(\sigma)(x) + w(x,y)$ with $w \in \mathbf{L}^{2,\perp}_p$ then

$$\int_{I\!R\times]0,2\pi[} r_1(x,y)\frac{1}{\cosh x}\mathrm{d}x\,\mathrm{d}y = \psi(\sigma).$$

Since $\psi(0) = 0$ and $\psi'(0) = 4\pi \neq 0$, ψ is a $\mathcal{C}^\infty$-diffeomorphism in a neighborhood of zero, so that for r small enough, σ is characterized by:

$$\sigma = \psi^{-1}\left(\int_{I\!R\times]0,2\pi[} r_1(x,y)\frac{1}{\cosh x}\mathrm{d}x\,\mathrm{d}y\right).$$

By subtraction we obtain then that w is characterized by $w = r - R(\sigma)$ which is automatically in $\mathbf{L}^{2,\perp}_p$.

The $\mathbf{H}^k_p$-regularity is a straightforward consequence of the previous decomposition since $R(\sigma)$ is smooth.

□

We aim to establish the equivalent formulation for Equation (2.12) in the new variables (σ, w). For a fixed $\bar\sigma \in I\!R$, $R(\bar\sigma)$ is a static solution for (2.12), so for all t we have

$$\begin{pmatrix} -\alpha & -1 \\ 1 & -\alpha \end{pmatrix} \mathcal{M}R(\sigma(t)) - h_a\mathcal{L}R(\sigma(t)) + \mathcal{F}(R(\sigma(t))) = 0.$$

Therefore plugging (4.32) in (2.11) we obtain

$$\partial_s R(\sigma)\frac{d\sigma}{dt} + \partial_t w = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \mathcal{M}w - h_a\mathcal{L}w + \mathcal{G}, \tag{4.34}$$

where $\mathcal{G} = \mathcal{F}(R(\sigma) + w) - \mathcal{F}(R(\sigma))$. We take the $\mathbf{L}^2_p$-inner product of (4.34) with $\dfrac{1}{4\pi\cosh x}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and using the same arguments as in Section 3.2, we obtain:

$$g(\sigma)\frac{d\sigma}{dt} = -h_a\mathcal{K}w + \tilde{G},$$

where

- $g(s) = \displaystyle\int_{x\in I\!R} \frac{1}{2\cosh x}\partial_s\rho(s)(x)\mathrm{d}x = 1 + \mathcal{O}(s)$, so that $g(s) \geq \frac{1}{2}$ for σ small enough,
- $\mathcal{K}w = \displaystyle\int_\Omega \frac{1}{4\pi\cosh x}\left(-\kappa\cos 2\theta\, w_2 - h_a(\alpha + \frac{1}{\alpha})\ell w_1 + 2h_a\tanh x\, w_2\right)\mathrm{d}x\,\mathrm{d}y,$
- $\tilde{G} = \displaystyle\int_\Omega \frac{1}{4\pi\cosh x}\mathcal{G}_1\mathrm{d}x\,\mathrm{d}y$ where $\mathcal{G}_1$ is the first component of $\mathcal{G}$.

Therefore we get

$$\frac{d\sigma}{dt} = \mathcal{K}_\sigma w + G \tag{4.35}$$

where $\mathcal{K}_\sigma w = \dfrac{\mathcal{K}w}{g(\sigma)}$ and $G = \dfrac{\tilde{G}}{g(\sigma)}$.

Writing in (4.34) that $\partial_s \rho(\sigma) = \dfrac{1}{\cosh x} + a(x,\sigma)$ where $a(x,\sigma) = \mathcal{O}(\sigma)$, using (4.35), we obtain by subtraction that :

$$\partial_t w = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \mathcal{M}w - h_a \mathcal{L}w - \frac{1}{\cosh x} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathcal{K}_\sigma w + \mathcal{H} \tag{4.36}$$

where

$$\mathcal{H} = \mathcal{G} - a(x,\sigma) \begin{pmatrix} \mathcal{K}_\sigma w \\ 0 \end{pmatrix} - \begin{pmatrix} \partial_s \rho(\sigma) G \\ 0 \end{pmatrix}. \tag{4.37}$$

In order to avoid the singularity of μ, we have to assume that $\|r\|_{L^\infty} \leq \frac{1}{2}$. In addition, we must assume that $\|r\|_{\mathbf{L}^2_p} \leq \xi_0$ to use Proposition 4.1, and that $|\sigma|$ small enough to be sure that $g(\sigma) \geq \frac{1}{2}$.
Therefore, using the Sobolev embedding of $\mathbf{H}^2_p$ into $L^\infty(I\!R^2)$, we introduce $\eta_0 > 0$ such that under the assumption:

$$|\sigma| \leq \eta_0 \quad \text{and} \quad \|w\|_{\mathbf{H}^2_p} \leq \eta_0, \tag{4.38}$$

then we have:

$$\|R(\sigma) + w\|_{L^\infty} \leq \frac{1}{2}, \quad \|R(\sigma) + w\|_{\mathbf{L}^2_p} \leq \frac{1}{2}, \quad \text{and} \quad g(\sigma) \geq \frac{1}{2}. \tag{4.39}$$

4.2 Nonlinear Stability

We fix an *a priori* bound on h_a: let h_a^{max} satisfying $0 < h_a^{max} < \frac{\alpha\kappa}{2}$ and let θ_{max} related to h_a^{max} by (1.10):

$$\theta_{max} = \frac{1}{2} \arcsin\left(\frac{2h_a^{max}}{\kappa}\right).$$

We introduce the constants α_1, α_2 and α_3 given by Proposition 3.1 with this θ_{max}, so that the norms equivalences in Proposition 3.1 are valid for all θ with $|\theta| \leq \theta_{max}$, *i.e.* for all h_a with $|h_a| \leq h_a^{max}$.

We aim to perform variational estimates on System (4.35)-(4.36). The right-hand side nonlinear terms are estimated in the following Proposition:

Proposition 4.2. *There exists a constant K such that for all h_a satisfying $|h_a| \leq h_a^{max}$, for all $\sigma \in I\!R$ satisfying $|\sigma| \leq \eta_0$, for all $w \in \mathbf{H}^3_p$ such that $\|w\|_{\mathbf{H}^2_p} \leq \eta_0$, then*

$$|\mathcal{K}_\sigma(w)| \leq K\|w\|_{\mathbf{H}^2_p}, \quad |G| \leq K\|w\|_{\mathbf{H}^2_p}, \quad \|\mathcal{H}\|_{\mathbf{L}^2_p} \leq K\left(|\sigma| + \|w\|_{\mathbf{H}^2_p}\right)\|w\|_{\mathbf{H}^2_p},$$

$$\text{and} \quad \|\mathcal{H}\|_{\mathbf{H}^1_p} \leq K\left(|\sigma| + \|w\|_{\mathbf{H}^2_p}\right)\|w\|_{\mathbf{H}^3_p}.$$

For the convenience of the reader the proof of this proposition is postponed to Section 5.

First step: H^1 estimates. Taking the inner product of (4.36) with $\mathcal{M}w$, since the first component of $\mathcal{M}w$ is orthogonal to $\dfrac{1}{\cosh x}$, we obtain:

$$\frac{1}{2}\frac{d}{dt}\langle w \mid \mathcal{M}w\rangle + \alpha\|\mathcal{M}w\|^2_{\mathbf{L}^2_p} = -h_a \langle \mathcal{L}w \mid \mathcal{M}w\rangle + \langle \mathcal{H} \mid \mathcal{M}w\rangle.$$

By Proposition 4.2 and by the norms equivalences established in Proposition 3.1, we obtain that there exists K_1 such that while $|\sigma| \leq \eta_0$ and $\|w\|_{\mathbf{H}^2_p} \leq \eta_0$, then

$$\|\mathcal{L}w\|_{\mathbf{L}^2_p} \leq K_1\|\mathcal{M}w\|_{\mathbf{L}^2_p} \quad \text{and} \quad \|\mathcal{H}\|_{\mathbf{L}^2_p} \leq K_1\left(|\sigma| + \|\mathcal{M}w\|_{\mathbf{L}^2_p}\right)\|\mathcal{M}w\|_{\mathbf{L}^2_p}.$$

So

$$\frac{1}{2}\frac{d}{dt}\langle w \mid \mathcal{M}w\rangle + \alpha\|\mathcal{M}w\|^2_{\mathbf{L}^2_p} \leq K_1\left(h_a + |\sigma| + \|\mathcal{M}w\|_{\mathbf{L}^2_p}\right)\|\mathcal{M}w\|^2_{\mathbf{L}^2_p}. \tag{4.40}$$

Second step: H^2 estimates. We will take the inner product of (4.36) with $\mathcal{M}^2 w$. Denoting by Y_1 and Y_2 the coordinates of $\mathcal{M}w$, we have on the one hand:

$$\begin{aligned}\left\langle \begin{pmatrix}-\alpha & -1\\ 1 & -\alpha\end{pmatrix}\mathcal{M}w \mid \mathcal{M}^2 w\right\rangle &= \left\langle \begin{pmatrix}-\alpha Y_1 - Y_2\\ Y_1 - \alpha Y_2\end{pmatrix} \mid \mathcal{M}Y\right\rangle,\\ &= -\alpha\langle Y \mid \mathcal{M}Y\rangle - \langle Y_2 \mid (\mathcal{M}Y)_1\rangle + \langle Y_1 \mid (\mathcal{M}Y)_2\rangle.\end{aligned}$$

Now,

$$\begin{aligned}-\langle Y_2 \mid (\mathcal{M}Y)_1\rangle &= -\left\langle Y_2 \mid (\frac{1}{\delta^2}L - \partial_{yy})Y_1 + 2\tanh x\,\partial_y Y_2\right\rangle,\\ &= -\left\langle (\frac{1}{\delta^2}L - \partial_{yy})Y_2 \mid Y_1\right\rangle,\end{aligned}$$

since $\frac{1}{\delta^2}L - \partial_{yy}$ is self-adjoint and by 2π-periodicity of Y_2, so that $\langle Y_2 \mid \tanh x\,\partial_y Y_2\rangle = 0$.

In addition, with the same arguments,

$$\begin{aligned}\langle Y_1 \mid (\mathcal{M}Y)_2\rangle &= \left\langle Y_1 \mid (\frac{1}{\delta^2}L - \partial_{yy})Y_2 + \kappa\cos 2\theta Y_2 - 2\tanh x\partial_y Y_1\right\rangle\\ &= \left\langle Y_1 \mid (\frac{1}{\delta^2}L - \partial_{yy})Y_2\right\rangle + \kappa\cos 2\theta\,\langle Y_1 \mid Y_2\rangle.\end{aligned}$$

Hence

$$\left\langle \begin{pmatrix}-\alpha & -1\\ 1 & -\alpha\end{pmatrix}\mathcal{M}w \mid \mathcal{M}^2 w\right\rangle = -\alpha\|\mathcal{M}^{\frac{1}{2}}Y\|^2_{\mathbf{L}^2_p} + \kappa\cos 2\theta\,\langle Y_1 \mid Y_2\rangle.$$

On the other hand,

$$\left\langle \frac{1}{\cosh x} \mid (\mathcal{M}Y)_1\right\rangle = \left\langle \frac{1}{\cosh x} \mid \frac{1}{\delta^2}LY_1 - \partial_{yy}Y_1 + 2\tanh x\,\partial_y Y_2\right\rangle = 0.$$

Therefore by taking the inner product of (4.36) with $\mathcal{M}^2 w$, we obtain:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathcal{M}w\|^2_{\mathbf{L}^2_p} + \alpha\|\mathcal{M}^{\frac{3}{2}}w\|^2_{\mathbf{L}^2_p} &= \kappa\cos 2\theta\,\langle(\mathcal{M}w)_1 \mid (\mathcal{M}w)_2\rangle - h_a\left\langle\mathcal{L}w \mid \mathcal{M}^2 w\right\rangle + \left\langle\mathcal{H} \mid \mathcal{M}^2 w\right\rangle\\ &\leq \kappa\|\mathcal{M}w\|^2_{\mathbf{L}^2_p} - h_a\left\langle\mathcal{M}^{\frac{1}{2}}\mathcal{L}w \mid \mathcal{M}^{\frac{3}{2}}w\right\rangle + \left\langle\mathcal{M}^{\frac{1}{2}}\mathcal{H} \mid \mathcal{M}^{\frac{3}{2}}w\right\rangle.\end{aligned}$$

Using Propositions 3.1 and 4.2, while $|\sigma| \leq \eta_0$ and $\|w\|_{\mathbf{H}^2_p} \leq \eta_0$, there exists K_2 such that

$$\|\mathcal{M}^{\frac{1}{2}}\mathcal{L}w\|_{\mathbf{L}^2_p} \leq K_2\|\mathcal{M}^{\frac{3}{2}}w\|_{\mathbf{L}^2_p}$$

and

$$\|\mathcal{M}^{\frac{1}{2}}\mathcal{H}\|_{\mathbf{L}^2_p} \leq K_2\left(|\sigma| + \|\mathcal{M}w\|_{\mathbf{L}^2_p}\right)\|\mathcal{M}^{\frac{3}{2}}w\|_{\mathbf{L}^2_p}.$$

Therefore we obtain that while $|\sigma| \leq \eta_0$ and $\|w\|_{\mathbf{H}^2_p} \leq \eta_0$,

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{M}w\|^2_{\mathbf{L}^2_p} + \alpha\|\mathcal{M}^{\frac{3}{2}}w\|^2_{\mathbf{L}^2_p} \leq \kappa\|\mathcal{M}w\|^2_{\mathbf{L}^2_p} + K_2\left(h_a + |\sigma| + \|\mathcal{M}w\|_{\mathbf{L}^2_p}\right)\|\mathcal{M}^{\frac{3}{2}}w\|^2_{\mathbf{L}^2_p}. \tag{4.41}$$

Adding up $2\kappa\times(4.40)$ and $\alpha\times(4.41)$ (so that the right-hand side term $\kappa\|\mathcal{M}w\|^2_{\mathbf{L}^2_p}$ in (4.41) is absorbed by the left-hand side of (4.40)), we obtain that while $|\sigma| \le \eta_0$ and $\|w\|_{\mathbf{H}^2_p} \le \eta_0$,

$$\frac{1}{2}\frac{d}{dt}\left(2\kappa \langle w \mid \mathcal{M}w\rangle + \alpha\|\mathcal{M}w\|^2_{\mathbf{L}^2_p}\right) + \alpha\kappa\|\mathcal{M}w\|^2_{\mathbf{L}^2_p} + \alpha^2\|\mathcal{M}^{\frac{3}{2}}w\|^2_{\mathbf{L}^2_p}$$
$$\le 2\kappa K_1\left(h_a + |\sigma| + \|\mathcal{M}w\|_{\mathbf{L}^2_p}\right)\|\mathcal{M}w\|^2_{\mathbf{L}^2_p} + \alpha K_2\left(h_a + |\sigma| + \|\mathcal{M}w\|_{\mathbf{L}^2_p}\right)\|\mathcal{M}^{\frac{3}{2}}w\|^2_{\mathbf{L}^2_p},$$

that is

$$\frac{1}{2}\frac{d}{dt}\left(2\kappa \langle w \mid \mathcal{M}w\rangle + \alpha\|\mathcal{M}w\|^2_{\mathbf{L}^2_p}\right) + \kappa\left(\alpha - 2K_1h_a - 2K_1|\sigma| - 2K_1\|\mathcal{M}w\|_{\mathbf{L}^2_p}\right)\|\mathcal{M}w\|^2_{\mathbf{L}^2_p}$$
$$+\alpha\left(\alpha - K_2h_a - K_2|\sigma| - K_2\|\mathcal{M}w\|_{\mathbf{L}^2_p}\right)\|\mathcal{M}^{\frac{3}{2}}w\|^2_{\mathbf{L}^2_p} \le 0.$$

We fix h_{max} by:

$$h_{max} = \min\left\{h_a^{max}, \frac{\alpha}{4K_1}, \frac{\alpha}{2K_2}\right\}.$$

From now on, we assume that

$$|h_a| \le h_{max}.$$

We obtain that while $|\sigma| \le \eta_0$ and $\|w\|_{\mathbf{H}^2_p} \le \eta_0$:

$$\frac{1}{2}\frac{d}{dt}\left(2\kappa \langle w \mid \mathcal{M}w\rangle + \alpha\|\mathcal{M}w\|^2_{\mathbf{L}^2_p}\right) + \kappa\left(\frac{\alpha}{2} - 2K_1|\sigma| - 2K_1\|\mathcal{M}w\|_{\mathbf{L}^2_p}\right)\|\mathcal{M}w\|^2_{\mathbf{L}^2_p}$$
$$+\alpha\left(\frac{\alpha}{2} - K_2|\sigma| - K_2\|\mathcal{M}w\|_{\mathbf{L}^2_p}\right)\|\mathcal{M}^{\frac{3}{2}}w\|^2_{\mathbf{L}^2_p} \le 0.$$

Third step: joint estimates for w and σ. We fix $\eta_1 = \min\left\{\eta_0, \frac{\alpha}{16K_1}, \frac{\alpha}{8K_2}\right\}$. While $|\sigma| \le \eta_1$ and $\|\mathcal{M}w\|_{\mathbf{L}^2_p} \le \eta_1$ then:

$$\frac{1}{2}\frac{d}{dt}\left(2\kappa \langle w \mid \mathcal{M}w\rangle + \alpha\|\mathcal{M}w\|^2_{\mathbf{L}^2_p}\right) + \frac{\alpha\kappa}{4}\|\mathcal{M}w\|^2_{\mathbf{L}^2_p} + \frac{\alpha^2}{4}\|\mathcal{M}^{\frac{3}{2}}w\|^2_{\mathbf{L}^2_p} \le 0.$$

Therefore, using Proposition 3.1, we obtain that while $|\sigma| \le \eta_1$ and $\|\mathcal{M}w\|_{\mathbf{L}^2_p} \le \eta_1$ then:

$$\frac{1}{2}\frac{d}{dt}\left(2\kappa\|\mathcal{M}^{\frac{1}{2}}w\|^2_{\mathbf{L}^2_p} + \alpha\|\mathcal{M}w\|^2_{\mathbf{L}^2_p}\right) + \frac{\alpha(\alpha_1)^2}{8}\left(2\kappa\|\mathcal{M}^{\frac{1}{2}}w\|^2_{\mathbf{L}^2_p} + \alpha\|\mathcal{M}w\|^2_{\mathbf{L}^2_p}\right) \le 0,$$

so that, by comparison lemma, while $|\sigma| \le \eta_1$ and $\|\mathcal{M}w\|_{\mathbf{L}^2_p} \le \eta_1$ then:

$$\left(2\kappa\|\mathcal{M}^{\frac{1}{2}}w(t)\|^2_{\mathbf{L}^2_p} + \alpha\|\mathcal{M}w(t)\|^2_{\mathbf{L}^2_p}\right) \le \left(2\kappa\|\mathcal{M}^{\frac{1}{2}}w(0)\|^2_{\mathbf{L}^2_p} + \alpha\|\mathcal{M}w(0)\|^2_{\mathbf{L}^2_p}\right)e^{-\frac{\alpha(\alpha_1)^2}{4}t}. \tag{4.42}$$

From (4.35) we have:

$$\left|\frac{d\sigma}{dt}\right| \le |\mathcal{K}_\sigma w| + |\mathcal{G}|, \tag{4.43}$$

hence, using Propositions 3.1 and 4.2, there exists a constant K_3 such that:

$$\left|\frac{d\sigma}{dt}\right| \le K_3\|\mathcal{M}w\|_{\mathbf{L}^2_p}.$$

Therefore, while $|\sigma| \le \eta_1$ and $\|\mathcal{M}w\|_{\mathbf{L}^2_p} \le \eta_1$, then

$$\left|\frac{d\sigma}{dt}\right| \le K_3\left(2\kappa\|\mathcal{M}^{\frac{1}{2}}w(0)\|^2_{\mathbf{L}^2_p} + \alpha\|\mathcal{M}w(0)\|^2_{\mathbf{L}^2_p}\right)^{\frac{1}{2}}\exp\left(-\frac{\alpha(\alpha_1)^2}{8}t\right), \tag{4.44}$$

so, by integration,

$$|\sigma(t)| \leq |\sigma(0)| + \frac{8K_3}{\alpha(\alpha_1)^2}\left(2\kappa\|\mathcal{M}^{\frac{1}{2}}w(0)\|^2_{\mathbf{L}^2_p} + \alpha\|\mathcal{M}w(0)\|^2_{\mathbf{L}^2_p}\right)^{\frac{1}{2}}. \tag{4.45}$$

End of the proof. Using Proposition 3.1, we introduce $\eta_2 > 0$ such that for any w, if $\|w(0)\|_{\mathbf{H}^2_p} \leq \eta_2$, then

$$\left(1 + \frac{8K_3}{\alpha(\alpha_1)^2}\right)\left(2\kappa\|\mathcal{M}^{\frac{1}{2}}w(0)\|^2_{\mathbf{L}^2_p} + \alpha\|\mathcal{M}w(0)\|^2_{\mathbf{L}^2_p}\right)^{\frac{1}{2}} \leq \frac{\eta_1}{4}.$$

We assume that $|\sigma(0)| \leq \frac{\eta_1}{4}$ and that $\|w(0)\|_{\mathbf{H}^2_p} \leq \eta_2$. Let us prove that for all $t \geq 0$, we have: $|\sigma(t)| < \eta$ and $\|\mathcal{M}(w(t))\|_{\mathbf{L}^2_p} < \eta_1$.

If not, since this property is obviously satisfied at $t = 0$, we introduce $t_1 > 0$ the first time in which this property fails. In particular, we have either $|\sigma(t_1)| = \eta_1$ or $\|\mathcal{M}(w(t_1))\|_{\mathbf{L}^2_p} = \eta_1$.

For all $t < t_1$, we have $|\sigma(t)| \leq \eta_1$ and $\|\mathcal{M}w(t)\|_{\mathbf{L}^2_p} \leq \eta_1$, so that Estimates (4.42) and (4.45) are valid on this interval. In particular, at $t = t_1$, by continuity we have:

$$\|\mathcal{M}w(t_1)\|_{\mathbf{L}^2_p} \leq \left(2\kappa\|\mathcal{M}^{\frac{1}{2}}w(0)\|^2_{\mathbf{L}^2_p} + \alpha\|\mathcal{M}w(0)\|^2_{\mathbf{L}^2_p}\right)^{\frac{1}{2}} \leq \frac{\eta_1}{4},$$

and

$$|\sigma(t_1)| \leq |\sigma(0)| + \frac{\eta_1}{4} \leq \frac{\eta_1}{2},$$

and this leads to a contradiction.

Therefore, if $\sigma(0) \leq \frac{\eta_1}{4}$ and $\|w(0)\|_{\mathbf{H}^2_p} \leq \eta_2$, then for all $t \geq 0$, (4.42) and (4.43) remain valid so that

- $\|w\|_{\mathbf{H}^2_p}$ remains small for all times,
- w tends to zero in $\mathbf{H}^2_p$ when t tends to $+\infty$,
- σ remains small for all times,
- since $\dfrac{d\sigma}{dt}$ is integrable on $I\!R^+$ by (4.44), σ tends to a limit σ_∞ when t tends to $+\infty$.

This concludes the proof of Theorem 1. $\square$

5 Estimate of the nonlinear terms

The aim of this part is to estimate the right-hand side terms of (4.35) and to obtain L^2 and H^1 estimates for the nonlinear terms in (4.36).

First we give the exact expression of $\mathcal{F}(r)$, the nonlinear term arising in Equation (2.12):

$$\begin{aligned}\mathcal{F} = \quad & F_1(r)(\partial_{xx}r) + F_2(r)(\partial_{yy}r) + F_3(r)(\partial_x r, \partial_x r) + F_4(r)(\partial_y r, \partial_y r) + F_5(x,r)(\partial_x r) \\ & + F_6(x,r)(\partial_y r) + F_7(x,r),\end{aligned} \tag{5.46}$$

with

- $F_1(r)(\partial_{xx}r) = \dfrac{1}{\delta^2}\begin{pmatrix} -\alpha r_1^2 & \mu(r)-\alpha r_1 r_2 \\ -\mu(r)-\alpha r_1 r_2 & -\alpha r_2^2 \end{pmatrix}\partial_{xx}r - \dfrac{1}{\delta^2}\begin{pmatrix} r_2+\alpha r_1+\alpha\mu(r)r_1 \\ \alpha r_2 - r_1 + \alpha\mu(r)r_2 \end{pmatrix}\mathrm{d}\mu(r)(\partial_{xx}r),$

- $F_2(r)(\partial_{yy}r) = \begin{pmatrix} -\alpha r_1^2 & \mu(r)-\alpha r_1 r_2 \\ -\mu(r)-\alpha r_1 r_2 & -\alpha r_2^2 \end{pmatrix}\partial_{yy}r - \begin{pmatrix} r_2+\alpha r_1+\alpha\mu(r)r_1 \\ \alpha r_2 - r_1 + \alpha\mu(r)r_2 \end{pmatrix}\mathrm{d}\mu(r)(\partial_{yy}r),$

- $F_3(r)(\partial_x r, \partial_x r) = -\dfrac{1}{\delta^2}\begin{pmatrix} r_2+\alpha r_1+\alpha\mu(r)r_1 \\ \alpha r_2 - r_1 + \alpha\mu(r)r_2 \end{pmatrix}\mathrm{d}^2\mu(r)(\partial_x r, \partial_x r),$

- $F_4(r)(\partial_y r, \partial_y r) = -\begin{pmatrix} r_2+\alpha r_1+\alpha\mu(r)r_1 \\ \alpha r_2 - r_1 + \alpha\mu(r)r_2 \end{pmatrix}\mathrm{d}^2\mu(r)(\partial_y r, \partial_y r),$

- $F_5(x,r)(\partial_x r) = -\dfrac{2}{\delta^2\cosh x}\begin{pmatrix} r_2+\alpha r_1+\alpha\mu(r)r_1 \\ \alpha r_2 - r_1 + \alpha\mu(r)r_2 \end{pmatrix}\partial_x r_1 + \dfrac{2}{\delta^2\cosh x}\begin{pmatrix} -\alpha+\alpha r_1^2 \\ \mu(r)+\alpha r_1 r_2 + 1 \end{pmatrix}\mathrm{d}\mu(r)(\partial_x r),$

- $F_6(x,r)(\partial_y r) = 2\tanh x\begin{pmatrix} \mu(r)-\alpha r_1 r_2 & \alpha r_1^2 \\ -\alpha r_2^2 & \mu(r)+\alpha r_1 r_2 \end{pmatrix}\partial_y r + \dfrac{2}{\cosh x}\begin{pmatrix} r_2+\alpha r_1+\alpha\mu(r)r_1 \\ \alpha r_2 - r_1 + \alpha\mu(r)r_2 \end{pmatrix}\partial_y r_2$

$$+\frac{2}{\cosh x}\begin{pmatrix} 1+\mu(r)-\alpha r_1 r_2 \\ \alpha - \alpha r_2^2 \end{pmatrix}\mathrm{d}\mu(r)(\partial_y r),$$

- $F_7(x,r) = \left(\dfrac{2}{\delta^2}\dfrac{\tanh x}{\cosh x}r_1 + \dfrac{h_a}{\alpha}\dfrac{1}{\cosh x}r_2 - \mu(r)(1-\dfrac{2}{\cosh^2 x}\dfrac{1}{\delta^2})\right)\begin{pmatrix} r_2+\alpha r_1+\alpha\mu(r)r_1 \\ -r_1+\alpha r_2+\alpha\mu(r)r_2 \end{pmatrix}$

$$-\alpha\left(1-\frac{2}{\cosh^2 x}\frac{1}{\delta^2}+h_a\tanh x\right)r\mu(r) + \frac{h_a}{\cosh x}\begin{pmatrix} \alpha \\ -1 \end{pmatrix}\mu(r)$$

$$+\left(h_a\frac{1}{\cosh x}+\kappa\sin^2\theta r_1 + \frac{h_a}{\alpha}\tanh x\, r_2\right)\begin{pmatrix} \alpha r_1^2 \\ \mu(r)+\alpha r_1 r_2 \end{pmatrix}$$

$$+\left(\frac{h_a}{\alpha}\frac{1}{\cosh x}+\kappa\cos^2\theta r_2 + \frac{h_a}{\alpha}\tanh x\, r_1 + \frac{h_a}{\alpha}\frac{1}{\cosh x}\mu(r)\right)\begin{pmatrix} \alpha r_1 r_2 - \mu(r) \\ \alpha r_2^2 \end{pmatrix}.$$

The term $\mathcal{G}$ arising in Equation (4.34) is defined by $\mathcal{G} = \mathcal{F}(R(\sigma)+w) - \mathcal{F}(R(\sigma))$, so that we have:

$$\mathcal{G} = G_1 + ... + G_7, \tag{5.47}$$

with

$$\begin{aligned}
G_1 = \;& F_1(R(\sigma)+w)(\partial_{xx}w) + \widetilde{F_1}(R(\sigma),w)(w)(\partial_{xx}R(\sigma)),\\
G_2 = \;& F_2(R(\sigma)+w)(\partial_{yy}w),\\
G_3 = \;& F_3(R(\sigma)+w)(\partial_x w, \partial_x w) + 2F_3(R(\sigma)+w)(\partial_x R(\sigma), \partial_x w)\\
& +\widetilde{F_3}(R(\sigma),w)(w)(\partial_x R(\sigma), \partial_x R(\sigma)),\\
G_4 = \;& F_4(R(\sigma)+w)(\partial_y w, \partial_y w),\\
G_5 = \;& F_5(x, R(\sigma)+w)(\partial_x w) + \widetilde{F_5}(x, R(\sigma), w)(w)(\partial_x R(\sigma)),\\
G_6 = \;& F_6(x, R(\sigma)+w)(\partial_y w),\\
G_7 = \;& \widetilde{F_7}(x, R(\sigma), w)(w).
\end{aligned}$$

Here,

$$\widetilde{F_i}\,(R(\sigma),w)=\int_0^1 \mathrm{d}_r F_i(R(\sigma)+s\,w)\mathrm{d}s \quad \text{ for } i\in\{1,3\},$$

and

$$\widetilde{F_i}\,(x,R(\sigma),w)=\int_0^1 \partial_r F_i(x,R(\sigma)+s\,w)\mathrm{d}s \quad \text{ for } i\in\{5,7\}.$$

Remark 5.1. *These terms are obtained by the Fundamental Theorem of Calculus writing for example that*

$$F_1(R(\sigma)+w)=F_1(R(\sigma))+\int_0^1 \mathrm{d}F_1(R(\sigma)+s\,w)(w)\mathrm{d}s=F_1(R(\sigma))+\widetilde{F_1}(R(\sigma),w)(w).$$

On the one hand we recall that

$$R(s)(x)=\begin{pmatrix}\rho(s)(x)\\ 0\end{pmatrix} \quad \text{with } \rho(s)(x)=-\frac{\tanh(x-s)}{\cosh x}+\frac{\tanh x}{\cosh(x-s)}.$$

So by direct calculations and estimates, we obtain that there exists a constant K such that if $|s|\leq 1$ then

$$\forall\, x\in I\!R,\quad |R(s)(x)|+|\partial_x R(s)(x)|+|\partial_{xx}R(s)(x)|+|\partial_{xxx}R(s)(x)|\leq \frac{K}{\cosh x}|s|. \tag{5.48}$$

On the other hand, since $\mu(r)=\sqrt{1-(r_1)^2-(r_2)^2}-1$, there exists a constant K such that if $|r|\leq \frac{1}{2}$ then

$$|\mu(r)|\leq K|r|^2,\quad |\mathrm{d}\mu(r)|\leq K|r| \quad \text{and} \quad |\mathrm{d}^2\mu(r)|+|\mathrm{d}^3\mu(r)|\leq K. \tag{5.49}$$

Let us now estimate each term of G defined by (5.47). In what follows we recall that we assume that $|\sigma|\leq\eta_0$ and $\|w\|_{\mathbf{H}^2_p}\leq\eta_0$, $\eta_0>0$ being small enough to ensure that $\|R(\sigma)+w\|_{L^\infty}\leq\frac{1}{2}$.

- **Estimate of G_1**

  We recall that

  $$G_1=F_1(R(\sigma)+w)(\partial_{xx}w)+\widetilde{F_1}(R(\sigma),w)(w)(\partial_{xx}R(\sigma)).$$

  Using (5.49) and the formulation of F_1 in (5.46), there exists K such that if $|r|\leq\frac{1}{2}$ then

  $$|F_1(r)|\leq K|r|^2,\quad |\mathrm{d}F_1(r)|\leq K|r| \quad \text{ and } |\mathrm{d}^2F_1(r)|\leq K. \tag{5.50}$$

  Since $\widetilde{F_1}(a,b)=\int_0^1 \mathrm{d}F_1(a+sb)ds$, we have also

  $$|\widetilde{F_1}(a,b)|\leq K(|a|+|b|) \quad \text{ and } |\mathrm{d}\widetilde{F_1}(a,b)|\leq K. \tag{5.51}$$

  Therefore we have

  $$\begin{aligned}|G_1|\leq\ & |F_1(R(\sigma)+w)|\,|\partial_{xx}w|+|\widetilde{F_1}(R(\sigma),w)|\,|w|\,|\partial_{xx}R(\sigma)|,\\ \leq\ & K|R(\sigma)+w|^2|\partial_{xx}w|+K(|R(\sigma)|+|w|)|w|\,|\partial_{xx}R(\sigma)|\\ & \quad \text{using (5.50) and (5.51)},\\ \leq\ & K(|\sigma|+|w|)(|\partial_{xx}w|+|w|) \quad \text{using (5.48)}.\end{aligned}$$

  Thus we get:

  $$\|G_1\|_{\mathbf{L}^2_p}\leq K(|\sigma|+\|w\|_{L^\infty})\|w\|_{\mathbf{H}^2_p}.$$

We estimate ∇G_1 on the following way:

$$\begin{aligned}|\nabla G_1| \leq\ & |\mathrm{d}F_1(R(\sigma)+w)|(|\partial_x R(\sigma)|+|\nabla w|)|\partial_{xx}w| + |F_1(R(\sigma)+w)||\nabla\partial_{xx}w| \\ & + |\mathrm{d}\widetilde{F_1}(R(\sigma),w)|(|\partial_x R(\sigma)|+|\nabla w|)|w||\partial_{xx}R(\sigma)| + |\widetilde{F_1}(R(\sigma),w)||\nabla w||\partial_{xx}R(\sigma)| \\ & + |\widetilde{F_1}(R(\sigma),w)||w||\partial_{xxx}R(\sigma)|,\end{aligned}$$

so using (5.48), (5.50) and (5.51), we obtain

$$\begin{aligned}|\nabla G_1| \leq\ & K(|\sigma|+|w|)(|\sigma|+|\nabla w|)|\partial_{xx}w| + K(|\sigma|+|w|)|\nabla\partial_{xx}w| \\ & + K(|\sigma|+|\nabla w|)|w||\sigma| + K(|\sigma|+|w|)|\nabla w||\sigma|.\end{aligned}$$

Thus,

$$\begin{aligned}\|\nabla G_1\|_{\mathbf{L}^2_p} \leq\ & K(|\sigma|+\|w\|_{L^\infty})\left(\|w\|_{\mathbf{L}^2_p}+\|\nabla w\|_{\mathbf{L}^2_p}+\|\partial_{xx}w\|_{\mathbf{L}^2_p}+\|\nabla\partial_{xx}w\|_{\mathbf{L}^2_p}\right) \\ & + K\|\nabla w\|_{L^4}\|\partial_{xx}w\|_{L^4}.\end{aligned}$$

We recall that in 2d, we have the following interpolation-type inequality:

$$\|v\|_{L^4(I\!R\times]0,2\pi[)} \leq C\|v\|^{\frac{1}{2}}_{L^2(I\!R\times]0,2\pi[)}\|v\|^{\frac{1}{2}}_{H^1(I\!R\times]0,2\pi[)}. \tag{5.52}$$

Using this inequality for the last term of the previous estimate of $\|\nabla G_1\|_{\mathbf{L}^2_p}$, we obtain that

$$\|\nabla G_1\|_{\mathbf{L}^2_p} \leq K\left(|\sigma|+\|w\|_{\mathbf{H}^2_p}\right)\|w\|_{\mathbf{H}^3_p}.$$

- **Estimate of G_2**

  We recall that $G_2 = F_2(R(\sigma)+w)(\partial_{yy}w)$, with $F_2 = \delta^2 F_1$ so with the same argument as for G_1 we obtain that there exists K such that if $|\sigma| \leq \eta_0$ and $\|w\|_{\mathbf{H}^2_p} \leq \eta_0$, then:

  $$\|G_2\|_{\mathbf{L}^2_p} \leq K(|\sigma|+\|w\|_{L^\infty})\|w\|_{\mathbf{H}^2_p},$$

  and

  $$\|\nabla G_2\|_{\mathbf{L}^2_p} \leq K\left(|\sigma|+\|w\|_{\mathbf{H}^2_p}\right)\|w\|_{\mathbf{H}^3_p}.$$

- **Estimate of G_3**

  We have

  $$G_3 = F_3(R(\sigma)+w)(\partial_x w,\partial_x w)+2F_3(R(\sigma)+w)(\partial_x R(\sigma),\partial_x w)+\widetilde{F}_3(R(\sigma),w)(w)(\partial_x R(\sigma),\partial_x R(\sigma)).$$

  Using (5.49) and the formulation of F_3 in (5.46), there exists K such that if $|r| \leq \frac{1}{2}$ then

  $$|F_3(r)| \leq K|r| \quad \text{ and } |\mathrm{d}F_3(r)|+|\mathrm{d}^2F_3| \leq K. \tag{5.53}$$

  Since $\widetilde{F_3}(a,b) = \int_0^1 \mathrm{d}F_3(a+sb)ds$, we have also

  $$|\widetilde{F_3}(a,b)|+|\mathrm{d}\widetilde{F_3}(a,b)| \leq K. \tag{5.54}$$

  Therefore we have, for $|\sigma| \leq \eta_0$ and $\|w\|_{\mathbf{H}^2_p} \leq \eta_0$,

$$
\begin{aligned}
|G_3| \leq\ & |F_3(R(\sigma)+w)||\partial_x w|^2 + 2|F_3(R(\sigma)+w)||\partial_x R(\sigma)||\partial_x w| \\
& + |\widetilde{F}_3(R(\sigma),w)||w||\partial_x R(\sigma)|^2, \\
\leq\ & K(|\sigma|+|w|)\left(|\partial_x w|^2 + |\sigma||\partial_x w|\right) + K|w||\sigma|^2,
\end{aligned}
$$

with (5.53) and (5.54). So,

$$
\begin{aligned}
\|G_3\|_{\mathbf{L}^2_p} \leq\ & K(|\sigma| + \|w\|_{L^\infty})\left(\|\partial_x w\|^2_{L^4} + |\sigma|\|\partial_x w\|_{\mathbf{L}^2_p}\right) + K|\sigma|^2\|w\|_{\mathbf{L}^2_p}, \\
\leq\ & K\left(|\sigma| + \|w\|_{\mathbf{H}^2_p}\right)\|w\|_{\mathbf{H}^2_p}.
\end{aligned}
$$

Let us estimate now ∇G_3:

$$
\begin{aligned}
|\nabla G_3| \leq\ & |\mathrm{d}F_3(R(\sigma)+w)|\left(|\partial_x R(\sigma)| + |\nabla w|\right)|\partial_x w|^2 + 2|F_3(R(\sigma)+w)||\nabla\partial_x w||\partial_x w| \\
& + 2|\mathrm{d}F_3(R(\sigma)+w)|\left(|\partial_x R(\sigma)| + |\nabla w|\right)|\partial_x R(\sigma)||\partial_x w| \\
& + 2|F_3(R(\sigma)+w)||\partial_{xx} R(\sigma)||\partial_x w| + 2|F_3(R(\sigma)+w)||\partial_x R(\sigma)||\nabla\partial_x w| \\
& + |\mathrm{d}\widetilde{F}_3(R(\sigma),w)|\left(|\partial_x R(\sigma)| + \nabla w|\right)|w||\partial_x R(\sigma)|^2 + |\widetilde{F}_3(R(\sigma),w)||\nabla w||\partial_x R(\sigma)|^2 \\
& + 2|\widetilde{F}_3(R(\sigma),w)||w||\partial_{xx} R(\sigma)||\partial_x R(\sigma)|, \\
\leq\ & K\left(|\sigma| + |\nabla w|\right)|\partial_x w|^2 + K(|\sigma|+|w|)|\nabla\partial_x w||\partial_x w| + K\left(|\sigma| + |\nabla w|\right)|\sigma||\partial_x w| \\
& + K(|\sigma|+|w|)|\sigma||\partial_x w| + K(|\sigma|+|w|)|\sigma||\nabla\partial_x w| + K\left(|\sigma| + |\nabla w|\right)|w||\sigma|^2 \\
& + K|\nabla w||\sigma|^2 + K(|\sigma|+|w|)|w||\sigma|^2 \quad \text{using (5.53) and (5.54).}
\end{aligned}
$$

So,

$$
\begin{aligned}
\|\nabla G_3\|_{\mathbf{L}^2_p} \leq\ & K|\sigma|\|\partial_x w\|^2_{L^4} + K\|\nabla w\|^2_{L^6} + K(|\sigma| + \|w\|_{L^\infty})\|\nabla\partial_x w\|_{L^4}\|\partial_x w\|_{L^4} \\
& + K|\sigma|^2\|\partial_x w\|_{L^2} + K\|\nabla w\|^2_{L^4}|\sigma| + K(|\sigma| + \|w\|_{L^\infty})|\sigma|\|\partial_x w\|_{L^2} \\
& + K(|\sigma| + \|w\|_{L^\infty})|\sigma|\|\nabla\partial_x w\|_{L^2} + K|\sigma|^3\|w\|_{\mathbf{L}^2_p} + K\|\nabla w\|_{L^4}\|w\|_{\mathbf{L}^2_p}|\sigma|^2 \\
& + K\|\nabla w\|_{L^2}|\sigma|^2 + K(|\sigma| + \|w\|_{L^\infty})\|w\|_{L^2}|\sigma|^2, \\
\leq\ & K\left(|\sigma| + \|w\|_{\mathbf{H}^2_p}\right)\|w\|_{\mathbf{H}^3_p} \quad \text{using Estimate (5.52).}
\end{aligned}
$$

- **Estimate of G_4**

  We have $G_4 = F_4(R(\sigma)+w)(\partial_y w, \partial_y w)$, with $F_4 = \delta^2 F_3$. So, this term is estimated as the first term of G_3 and we obtain that

$$
\|G_4\|_{\mathbf{L}^2_p} \leq K(|\sigma| + \|w\|_{\mathbf{H}^2_p})\|w\|_{\mathbf{H}^2_p} \quad \text{and} \quad \|\nabla G_4\|_{\mathbf{L}^2_p} \leq K\left(|\sigma| + \|w\|_{\mathbf{H}^2_p}\right)\|w\|_{\mathbf{H}^3_p}.
$$

- **Estimate of G_5**

  This term writes

$$
G_5 = F_5(x, R(\sigma)+w)(\partial_x w) + \widetilde{F_5}(x, R(\sigma), w)(w)(\partial_x R(\sigma)),
$$

where F_5 and $\widetilde{F_5}$ satisfy that there exists K such that for $|r| \leq \frac{1}{2}$, for $x \in I\!R$, we have:

$$|F_5(x,r)| + |\partial_x F_5(x,r)| \leq K|r|, \quad |\partial_r F_5(x,r)| \leq K, \tag{5.55}$$

and

$$|\widetilde{F}_5(x,a,b)| + |\partial_x \widetilde{F}_5(x,a,b)| + |\partial_r \widetilde{F}_5(x,a,b)| \leq K. \tag{5.56}$$

We have

$$\begin{aligned} |G_5| \leq & \quad |F_5(x, R(\sigma)+w)||\partial_x w| + |\widetilde{F_5}(x, R(\sigma), w)||w||\partial_x R(\sigma)|, \\ \leq & \quad K(|\sigma| + |w|)|\partial_x w| + K|\sigma||w|, \end{aligned}$$

thus

$$\|G_5\|_{\mathbf{L}^2_p} \leq K(|\sigma| + \|w\|_{L^\infty})\|w\|_{\mathbf{H}^1_p}.$$

In addition,

$$\begin{aligned} |\nabla G_5| \leq & \quad |\mathrm{d}F_5(x, R(\sigma)+w)| \left(|\partial_x R(\sigma)| + |\nabla w|\right)|\partial_x w| + |F_5(x, R(\sigma)+w)||\nabla \partial_x w| \\ & + \left(|\partial_x \widetilde{F_5}(x, R(\sigma), w)| + |\partial_a \widetilde{F_5}(x, R(\sigma), w)||\partial_x R(\sigma)|\right)|w||\partial_x R(\sigma)| \\ & + |\partial_b \widetilde{F_5}(x, R(\sigma), w)||\nabla w||w||\partial_x R(\sigma)| \\ & + |\widetilde{F_5}(x, R(\sigma), w)||\nabla w||\partial_x R(\sigma)| + |\widetilde{F_5}(x, R(\sigma), w)||w||\partial_{xx} R(\sigma)|, \\ \leq & \quad K(|\sigma| + |\nabla w|)|\partial_x w| + K(|\sigma| + |w|)|\nabla \partial_x w| + K(1+|\sigma|)|w||\sigma| \\ & K|\nabla w||w||\sigma| + K|\nabla w||\sigma| + K|w||\sigma|. \end{aligned}$$

So,

$$\begin{aligned} \|\nabla G_5\|_{\mathbf{L}^2_p} \leq & \quad K(|\sigma| + \|w\|_{L^\infty})\left(\|w\|_{\mathbf{L}^2_p} + \|\nabla w\|_{\mathbf{L}^2_p} + \|\nabla \partial_x w\|_{\mathbf{L}^2_p}\right) + \|\nabla w\|^2_{L^4}, \\ \leq & \quad K(|\sigma| + \|w\|_{\mathbf{H}^2_p})\|w\|_{\mathbf{H}^2_p}. \end{aligned}$$

- **Estimate of G_6**

We recall that

$$G_6 = F_6(x, R(\sigma)+w)(\partial_y w),$$

with

$$|F_6(x,r)| + |\partial_x F_6(x,r)| \leq K|r| \quad \text{ and } \quad |\partial_r F_6| \leq K. \tag{5.57}$$

Hence

$$\|G_6\|_{\mathbf{L}^2_p} \leq \|F_6(x, R(\sigma)+w)\|_{L^\infty}\|\partial_y w\|_{\mathbf{L}^2_p} \leq K(|\sigma| + \|w\|_{L^\infty})\|w\|_{\mathbf{H}^1_p}.$$

In addition,

$$\begin{aligned} |\nabla G_6| \leq & \quad |\partial_x F_6(x, R(\sigma)+w)||\partial_y w| + |\partial_r F_6(x, R(\sigma)+w)|\left(|\partial_x R(\sigma)| + |\nabla w|\right)|\partial_y w| \\ & + |F_6(x, R(\sigma)+w)||\nabla \partial_y w|, \\ \leq & \quad K(|\sigma| + |\nabla w|)(|\partial_y w| + |\nabla \partial_y w|) + K|\nabla w||\partial_y w|. \end{aligned}$$

So

$$\|\nabla G_6\|_{\mathbf{L}^2_p} \leq K(|\sigma| + \|w\|_{\mathbf{H}^2_p})\|w\|_{\mathbf{H}^2_p}.$$

- **Estimate of G_7**

  This last term is given by
  $$G_7 = \widetilde{F_7}(x, R(\sigma), w)(w),$$
  where, for $|a| + |b| \leq \frac{1}{2}$,
  $$|\widetilde{F_7}(x,a,b)| + |\partial_x \widetilde{F_7}(x,a,b)| \leq K(|a|+|b|) \quad \text{and } |\partial_a \widetilde{F_7}(x,a,b)| + |\partial_b \widetilde{F_7}(x,a,b)| \leq K.$$
  Therefore
  $$\|G_7\|_{\mathbf{L}^2_p} \leq \|\widetilde{F_7}(x, R(\sigma), w)\|_{L^\infty} \|w\|_{\mathbf{L}^2_p} \leq K(|\sigma| + \|w\|_{L^\infty})\|w\|_{\mathbf{L}^2_p}.$$
  In addition,
  $$\begin{aligned} |\nabla G_7| \leq & \quad |\partial_x \widetilde{F_7}(x, R(\sigma), w)||w| + |\partial_a \widetilde{F_7}(x, R(\sigma), w)||\partial_x R(\sigma)||w| \\ & + |\partial_b \widetilde{F_7}(x, R(\sigma), w)||\nabla w||w| + |\widetilde{F_7}(x, R(\sigma), w)||\nabla w| \\ \leq & \quad K(|\sigma| + |w|)|w| + K|\sigma||w| + K|\nabla w||w| + K(|\sigma| + |w|)|\nabla w|. \end{aligned}$$
  So
  $$\|\nabla G_7\|_{\mathbf{L}^2_p} \leq K(|\sigma| + \|w\|_{L^\infty})\|w\|_{\mathbf{H}^1_p}.$$

Therefore we obtain that
$$\|\mathcal{G}\|_{\mathbf{L}^2_p} \leq K\left(|\sigma| + \|w\|_{\mathbf{H}^2_p}\right)\|w\|_{\mathbf{H}^2_p} \quad \text{and} \quad \|\mathcal{G}\|_{\mathbf{H}^1_p} \leq K\left(|\sigma| + \|w\|_{\mathbf{H}^2_p}\right)\|w\|_{\mathbf{H}^3_p}. \tag{5.58}$$
Since $\tilde{\mathcal{G}} = \left\langle \mathcal{G}_1 \mid \dfrac{1}{4\pi \cosh x} \right\rangle$, where $\mathcal{G}_1$ is the first component of $\mathcal{G}$, we have
$$|\tilde{G}| \leq 2\pi \|\mathcal{G}\|_{\mathbf{L}^2_p} \leq K\left(|\sigma| + \|w\|_{\mathbf{H}^2_p}\right)\|w\|_{\mathbf{H}^2_p}. \tag{5.59}$$
So, using that $g(\sigma)$ is bounded by below by $\frac{1}{2}$, we obtain that there exists K such that if $|\sigma| \leq \eta_0$ and $\|w\|_{\mathbf{H}^2_p} \leq \eta_0$,
$$|G| \leq K\left(|\sigma| + \|w\|_{\mathbf{H}^2_p}\right)\|w\|_{\mathbf{H}^2_p}. \tag{5.60}$$
From the expression of $\mathcal{K}_\sigma(w)$ we have
$$|\mathcal{K}_\sigma(w)| \leq C\left(\|w_2\|_{\mathbf{L}^2_p} + \|\ell w_1\|_{\mathbf{L}^2_p}\right). \tag{5.61}$$
Thus, as $|a(x,\sigma)| + |\partial_x a(x,\sigma)| \leq K|\sigma|$, using the previously-obtained estimates we obtain that
$$\|\mathcal{H}\|_{\mathbf{L}^2_p} \leq K\left(|\sigma| + \|w\|_{\mathbf{H}^2_p}\right)\|w\|_{\mathbf{H}^2_p},$$
and
$$\|\mathcal{H}\|_{\mathbf{H}^1_p} \leq K\left(|\sigma| + \|w\|_{\mathbf{H}^2_p}\right)\|w\|_{\mathbf{H}^3_p}.$$
Concerning the right-hand side terms in Equation (4.35), from (5.61), we get:
$$|\mathcal{K}_\sigma(w)| \leq K\|w\|_{\mathbf{H}^2_p},$$
and from (5.60), since $|\sigma| \leq \eta_0$ and $\|w\|_{\mathbf{H}^2_p} \leq \eta_0$, we obtain that there exists K such that
$$|G| \leq K\|w\|_{\mathbf{H}^2_p}.$$
This concludes the proof of Proposition 4.2. □

References

[1] Allwood DA, Xiong G, Faulkner CC, Atkinson D, Petit D and Cowburn RP, *Magnetic Domain-wall Logic*, Science **309** (2005), 1688–1692.

[2] François Alouges and Alain Soyeur, *On global weak solutions for Landau-Lifshitz equations: existence and nonuniqueness,* Nonlinear Anal. **18** (1992), no. 11, 1071–1084.

[3] William F. Brown, Micromagnetics, Wiley, New York (1963).

[4] Gilles Carbou, *Stability of static walls for a three-dimensional model of ferromagnetic material,* J. Math. Pures Appl. **93** (2010), no. 2, 183–203.

[5] Gilles Carbou, *Thin Layers in Micromagnetism*, Math. Models and Meth. in Applied Sciences **11** (2001), 1529–1546.

[6] Gilles Carbou and Pierre Fabrie, *Time average in Micromagnetism*, Journal of Differential Equations **147** (1998), 383–409.

[7] Gilles Carbou and Pierre Fabrie, *Regular Solutions for Landau-Lifschitz Equation in a Bounded Domain*, Differential and Integral Equations **14** (2001), 213-229.

[8] Gilles Carbou and Pierre Fabrie, *Regular solutions for Landau-Lifschitz equation in* $\mathbb{R}^3$, Commun. Appl. Anal. **5** (2001), no. 1, 17–30.

[9] Gilles Carbou and Rida Jizzini, *Domain walls dynamics in a nanowire subject to an electric current*, J. Differential Equations **258** (2015), no. 8, 2941–2965.

[10] Gilles Carbou and Stéphane Labbé, *Stability for static walls in ferromagnetic nanowires,* Discrete Contin. Dyn. Syst. Ser. B **6** (2006), no. 2, 273–290.

[11] Gilles Carbou and Sthéphane Labbé, *Stabilization of Walls for Nano-Wires of Finite Length.* ESAIM Control Optim. Calc. Var. **18** (2012), no. 1, 1–21.

[12] Gilles Carbou, Stéphane Labbé and Emmanuel Trélat, *Control of travelling walls in a ferromagnetic nanowire,* Discrete Contin. Dyn. Syst. Ser. S **1** (2008), no. 1, 51–59.

[13] Shijin Ding, Boling Guo, Junyu Lin and Ming Zeng, *Global existence of weak solutions for Landau-Lifshitz-Maxwell equations, Discrete Contin. Dyn. Syst.* **17** (2007), no. 4, 867–890.

[14] G. Gioia and R. D. James, *Micromagnetics of very thin films*, R. Soc. Lond. Proc. Ser A **453** (1997), 213–223.

[15] Boling Guo and Fengqiu Su, *Global weak solution for the Landau-Lifshitz-Maxwell equation in three space dimensions, J. Math. Anal. Appl.* **211** (1997), no. 1, 326–346.

[16] Todd Kapitula, *On the stability of travelling waves in weighted* L^∞ *spaces*, J. Differential Equations **112** (1994), no. 1, 179–215.

[17] Todd Kapitula, *Multidimensional stability of planar travelling waves*, Trans. Amer. Math. Soc. **349** (1997), no. 1, 257–269.

[18] Stéphane Labbé, Simulation numérique du comportement hyperfréquence des matériaux ferromagnétiques. Thèse de l'Université Paris 13 (1998).

[19] Stuart S.P. Parkin, Masamitsu Hayashi, Luc Thomas, *Magnetic domain-wall racetrack memory*, Science **320** (2008), 190–194.

[20] Keisuke Takasao, *Stability of travelling wave solutions for the Landau-Lifshitz equation*, Hiroshima Math. J. **41** (2011), no. 3, 367–388.

[21] André Thiaville and Yoshinobu Nakatani, *Domain wall dynamics in nanowires and nanostrips*, Spin dynamics in confined magnetic structures III, B. Hillebrands, A. Thiaville Eds. Topics in Applied Physics **101**, pp. 161-206 (Springer, 2006)

[22] Augusto Visintin, *On Landau Lifschitz equation for ferromagnetism,* Japan Journal of Applied Mathematics **1** (1985), no. 2, 69-84.

[23] Laurence R. Walker, Bell Telephone Laboratories Memorandum, 1956 (unpublished). An account of this work is to be found in J.F. Dillon, Jr., Magnetism Vol III, edited by G.T. Rado and H. Subl, Academic, New York, 1963.

[24] Ming Yan, Christian Andreas, Attila Kákay, Felipe García-Sánchez and Riccardo Hertel, *Fast domain wall dynamics in magnetic nanotubes: Suppression of Walker breakdown and Cherenkov-like spin wave emission*, Appl. Phys. Lett. **99** (2011)