Domain walls dynamics for one dimensional models of Ferromagnetic Nanowires

Gilles Carbou
Laboratoire de Mathématiques et de leurs Applications - Pau
UMR CNRS 5142
Bâtiment IPRA - Université de Pau et des Pays de l'Adour
Avenue de l'Université - BP 1155
64013 PAU CEDEX

1 Introduction

Ferromagnetic materials play a more and more important role for the storage of digital information. In particular, one of the most promising innovating technology to ensure a cheap storage with rapid access time to the information is the use of ferromagnetic nanowires in so called racetrack memories (see [47]). In such devices, the bits are encoded as magnetic domains separated by domains walls along the wire. One can obtain truly three dimensional devices by using U-shaped nanowires normal to the plane of a silicon wafer (see [47] fig 1.A and 1.E). This is a promising way to increase the storage capacity in cheap and fast devices. In racetrack memories, reading the information is obtained by shifting the magnetic configuration with an electric current: the domain walls are moved along the wire to reading or writing elements. The displacement of domains walls is also possible using non uniform local magnetic field, even if this technic is no more used because of its cost and complexity. From the mathematical point of view, the description of domain walls dynamics in ferromagnetic nanowires is a recent topic. In this paper, we review several contributions concerning straight nanowires of ferromagnetic material submitted to a magnetic field or an electric current. To start with, we derive in the second part a one-dimensional model of ferromagnetic nanowires from the three dimensional model by asymptotic analysis when the diameter of the wire tends to zero. In the third part, we describe the domain walls by exact solutions for the obtained one dimensional model. We prove stability results for one wall configurations in an infinite nanowire. For finite wires or for periodic configurations in infinite wires, one can prove that the exact solutions are unstable. In addition, we remark that exact solutions cannot describe realistic configurations with several walls located at arbitrary positions. These configurations can be described by quasi-solutions which metastability is studied in Part 4. We conclude the paper with a list of open problems.

2 Model of Ferromagnetic Nanowires

2.1 Three-Dimensional Model

Ferromagnetic materials (like magnets) are characterized by a spontaneous magnetization described by a vector field m, called magnetic moment and defined on the ferromagnetic domain Ω (see [10], [32] and [43] for more details). The magnetic moment links the magnetic induction B and the magnetic field H by the constitutive relation:

$$B = H + \overline{m}, \tag{2.1}$$

where B and H are defined on the whole space $\mathbb{R}^3$ and where $\overline{m}$ denotes the extension of m by zero outside Ω. At low temperature, the material is said to be saturated, that is the norm of m is constant. After renormalization, the saturation constraint writes:

$$|m(t,x)| = 1 \text{ for all } (t,x) \in \mathbb{R}^+ \times \Omega. \tag{2.2}$$

The micromagnetism energy associated to a configuration m is given by:

$$\mathcal{E}_{mic}(m) = \mathcal{E}_{exch} + \mathcal{E}_{dem} + \mathcal{E}_{Zee}, \tag{2.3}$$

where

- the exchange energy is derived from the Eisenberg model of interaction between two spins:

$$\mathcal{E}_{exch} = A \int_\Omega |\nabla m|^2 \, dX,$$

where the exchange coefficient A depends on the material.

- The demagnetizing energy measures the energy of the magnetic field $H_d(m)$ induced by the magnetization m. This field is given writing the static Maxwell equations and the law of Faraday div $B = 0$, that is $H_d(m)$ is obtained from m by the relations:

$$\text{curl } H_d(m) = 0 \text{ and div } (H_d(m) + \overline{m}) = 0 \text{ in } \mathbb{R}^3. \tag{2.4}$$

The demagnetizing energy is given by:

$$\mathcal{E}_{dem} = \int_{\mathbb{R}^3} |H_d(m)|^2 \, dX = -\int_\Omega m \cdot H_d(m) \, dX. \tag{2.5}$$

- The Zeeman energy reflects the effects of an applied magnetic field H_a on the magnetization distribution:

$$\mathcal{E}_{Zee} = -2 \int_\Omega H_a \cdot m \, dX. \tag{2.6}$$

The static configurations of the magnetization are the local minimizers of the energy (2.3) under the saturation constraint (2.1), *i.e.* they are obtained by minimizing $\mathcal{E}(m)$ for $m \in H^1(\Omega; S^2)$ with:

$$H^1(\Omega; S^2) = \left\{ m \in H^1(\Omega; \mathbb{R}^3), \, |m| = 1 \text{ a.e.} \right\}.$$

The partial regularity of these minimizers in studied in [7], [11] and [33].
The dynamics for the magnetization is described by the Landau-Lifschitz equation:

$$\frac{\partial m}{\partial t} = -m \times H_e - m \times (m \times H_e), \tag{2.7}$$

where the effective field H_e is derived from the micromagnetic energy:

$$H_e = -\frac{1}{2} \partial_m \mathcal{E}_{mic} = A \Delta m + H_d(m) + H_a. \tag{2.8}$$

The Landau-Lifschitz system (2.7)-(2.8) is a parabolic type problem and the natural condition on $\partial\Omega$ is the Neumann homogeneous boundary condition:

$$\frac{\partial m}{\partial \nu} = 0 \text{ on } \partial\Omega, \text{ where } \nu \text{ is the unit exterior normal vector to the boundary.} \tag{2.9}$$

We remark that, at least formally, the Landau-Lifschitz equation preserves the saturation constraint (2.2) since the left hand side term of the equation is orthogonal to m. It tends to align m with H_e and to decrease the micromagnetism energy.

Existence and uniqueness of local in time strong solutions for (2.7) via a Galerkin approximation and variational estimates are established in [16] and [17].

From the numerical point of view, the main difficulties are the conservation of the saturation constraint, the quasilinear character of the equations, and the computation of the non local demagnetizing field. The interested reader can consult [4, 8, 37, 38, 39, 40, 41, 44].

Concerning weak solutions, we deal with the following form of the Laudau-Lifschitz equation, called the Landau-Lifschitz-Gilbert equation. If m is sufficiently smooth, then equation (2.7) is equivalent to the system:

$$\frac{\partial m}{\partial t} - m \times \frac{\partial m}{\partial t} = -2m \times H_e. \tag{2.10}$$

This form is more convenient to obtain a weak formulation for the Landau-Lifschitz system, using that

$$m \times \Delta m = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left(m \times \frac{\partial m}{\partial x_i} \right).$$

In addition, taking the scalar product of (2.10) with $\frac{\partial m}{\partial t} - 2H_e$, one can formally obtain the energy decreasing formula:

$$\frac{d\mathcal{E}_{mic}}{dt} + \int_\Omega |\frac{\partial m}{\partial t}|^2 = 2\int_\Omega m\partial_t H_a.$$

Existence of global in time weak solutions for (2.7) is tackled in several papers : [3], [15], [30], [38] and [55]. More precisely, we have the following result:

Proposition 2.1. *Let $m_0 \in H^1(\Omega; S^2)$. There exists $m : \mathbb{R}^+ \times \Omega \to \mathbb{R}^3$ satisfying:*

- $m \in L^\infty(\mathbb{R}^+; H^1(\Omega))$ *and* $\frac{\partial m}{\partial t} \in L^2(\mathbb{R}^+; L^2(\Omega))$,
- $|m(t,x)| = 1$ *a.e. (saturation constraint),*
- *for all* $\Psi \in \mathcal{C}_c^\infty(\mathbb{R}^+; H^1(\Omega))$,

$$\int_{\mathbb{R}^+\times\Omega} \left(\frac{\partial m}{\partial t} - m \times \frac{\partial m}{\partial t} \right) \cdot \Psi = 2A \int_{\mathbb{R}^+\times\Omega} \sum_{i=1}^{3} \left(m \times \frac{\partial m}{\partial x_i} \right) \cdot \frac{\partial \Psi}{\partial x_i} - 2\int_{\mathbb{R}^+\times\Omega} (\mathcal{H}(m) + H_a) \cdot \Psi, \tag{2.11}$$

- $m(0,\cdot) = m_0$ *in the trace sense,*
- *for all $t > 0$, we have the following energy inequality:*

$$\mathcal{E}_{mic}(m(t)) + \int_0^t \left| \frac{\partial m}{\partial t} \right|^2 \leq \mathcal{E}_{mic}(m(0)) + 2\int_0^t \int_\Omega |\partial_t H_a|.$$

Remark 2.1. *The saturation constraint and the energy inequality are obtained by construction. They can not be proved directly from the equation for weak solutions. For instance, if one does not know* a priori *that m is in $L^\infty(\mathbb{R}^+ \times \Omega)$, it is not possible to take m as a test function in the weak formulation (2.11) to derive the saturation constraint.*

Remark 2.2. *For an effective field reduced to $A\Delta m$, the non uniqueness of weak solutions is proved in [3] for particular initial data. This non uniqueness remains an open problem for the complete model.*

If the following section, we derive from this three-dimensional model an asymptotic one-dimensional model for wires by taking the limit of the system when the diameter of the wire tends to zero.

2.2 Asymptotic Model for Nanowires

We denote by (e_1, e_2, e_3) the canonical basis of $\mathbb{R}^3$, and we denote by (x, y, z) the coordinates in $\mathbb{R}^3$. We consider a cylindrical magnetic domain $\Omega_\eta = [0, L] \times B_2(0, \eta)$, where $B_2(0, \eta)$ is the ball of radius η and center 0 in $\mathbb{R}^2$. In addition, we assume that the applied field $H_a : \mathbb{R}_t \times \Omega_\eta \to \mathbb{R}^3$ does not depend on the transverse variable and is polarized along e_1:

$$H_a(t, x, y, z) = h_a(t, x)e_1.$$

Let $m_0 \in H^1([0, L]; S^2)$. For $(x, y, z) \in \Omega_\eta$, we denote $m_0^\eta(x, y, z) = m_0(x)$. We consider m^η a weak solution of the Landau-Lifschitz-Gilbert equation (2.10) on the domain Ω_η with initial data

m_0^η given by Proposition 2.1. We will deal with a rescaled version of the system by introducing M^η and $\mathcal{H}^\eta$ given by:

$$M^\eta(t,x,y,z) = m^\eta(t,x,\eta y,\eta z) \text{ and } \mathcal{H}^\eta(t,x,y,z) = (\mathcal{H}(m^\eta))(t,x,\eta y,\eta z) \text{ for } (x,y,z)\in\Omega_1.$$

The rescaled energy writes:

$$\begin{aligned}\mathcal{E}^\eta(M^\eta) = & \frac{1}{\eta^2}\mathcal{E}_{mic}(m^\eta)\\ = & A\int_{\Omega_1}\left|\frac{\partial M^\eta}{\partial x}\right|^2 + \frac{A}{\eta^2}\int_{\Omega_1}\left(\left|\frac{\partial M^\eta}{\partial y}\right|^2+\left|\frac{\partial M^\eta}{\partial z}\right|^2\right) + \int_{\mathbb{R}^3}|\mathcal{H}^\eta|^2 - 2\int_{\Omega_1} M^\eta\cdot h_a e_1.\end{aligned}$$

Therefore M^η satisfies the following properties:

- $M^\eta\in L^\infty(\mathbb{R}^+;H^1(\Omega_1))$ and $\dfrac{\partial M^\eta}{\partial t}\in L^2(\mathbb{R}^+;L^2(\Omega_1))$,
- $|M^\eta(t,x,y,z)| = 1$ *a.e.*,
- for all $\Psi\in\mathcal{C}_c^\infty(\mathbb{R}^+;H^1(\Omega_1))$,

$$\begin{aligned}\int_{\mathbb{R}^+\times\Omega_1}\left(\frac{\partial M^\eta}{\partial t} - M^\eta\times\frac{\partial M^\eta}{\partial t}\right)\cdot\Psi = 2A\int_{\mathbb{R}^+\times\Omega_1}\left(M^\eta\times\frac{\partial M^\eta}{\partial x}\right)\cdot\frac{\partial\Psi}{\partial x}\\ +\frac{2}{\eta^2}\int_{\mathbb{R}^+\times\Omega_1}\sum_{i=2}^3\left(M^\eta\times\frac{\partial M^\eta}{\partial x_i}\right)\cdot\frac{\partial\Psi}{\partial x_i} - 2\int_{\mathbb{R}^+\times\Omega_1}(M^\eta\times(\mathcal{H}^\eta + h_a\,e_1))\cdot\Psi,\end{aligned}\tag{2.12}$$

- $M^\eta(0,\cdot) = m_0$ in the trace sense,
- for all $t>0$, we have the following energy inequality:

$$\mathcal{E}^\eta(M^\eta(t)) + \int_0^t\int_{\Omega_1}\left|\frac{\partial M^\eta}{\partial t}\right|^2 \le \mathcal{E}^\eta(M^\eta(0)) + 2\int_0^t\int_{\Omega_1}|\partial_t h_a|.$$

Since the initial data $M^\eta(0)$ does not depend on y and z, with reasonable assumptions on the applied field h_a, the right hand side of the energy inequality is uniformly bounded when η tend to zero. Therefore there exists C such that for all T and all η in a neighborhood of 0, we have:

- $\|M^\eta\|_{L^\infty(0,T;H^1(\Omega_1))}\le C$,
- $\left\|\dfrac{\partial M^\eta}{\partial y}\right\|_{L^\infty(0,T;L^2(\Omega_1))} + \left\|\dfrac{\partial M^\eta}{\partial z}\right\|_{L^\infty(0,T;L^2(\Omega_1))}\le C\eta$,
- $\left\|\dfrac{\partial M^\eta}{\partial t}\right\|_{L^2(0,T;L^2(\Omega_1))}\le C$,
- $\|\mathcal{H}^\eta\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}\le C$.

So we can extract subsequences till denoted by M^η and $\mathcal{H}^\eta$ such that:

- $M^\eta\rightharpoonup M$ in $L^\infty(0,T;H^1(\Omega_1))$ weak *,
- $\dfrac{\partial M^\eta}{\partial y}$ and $\dfrac{\partial M^\eta}{\partial z}$ tend to zero in $L^\infty(0,T;L^2(\Omega_1))$, so that M only depends on $(t,x)\in\mathbb{R}^+\times[0,L]$ and does not depend on y and z,
- $\dfrac{\partial M^\eta}{\partial t}\rightharpoonup\dfrac{\partial M}{\partial t}$ in $L^2(0,T;L^2(\Omega_1))$ weak,

- $\mathcal{H}^\eta \rightharpoonup \mathcal{H}^0$ in $L^\infty(0,T;L^2(\mathbb{R}^3))$ weak *.

Using the Aubin-Simon compactness lemma, $M^\eta \to M$ in $L^\infty(0,T;L^p(\Omega_1))$ strongly for $p<6$, and, by extracting a subsequence, we can assume that $M^\eta \to M$ almost everywhere in $[0,T]\times\Omega_1$, so that M satisfies the saturation constraint

$$|M(t,x)| = 1 \ a.e. \text{ in } \mathbb{R}^+ \times [0,L].$$

Concerning the energy inequality, we define the reduced energy $\tilde{\mathcal{E}}$ by:

$$\tilde{\mathcal{E}}(M^\eta) := A\int_{\Omega_1}\left|\frac{\partial M^\eta}{\partial x}\right|^2 + \int_{\mathbb{R}^3}|\mathcal{H}^\eta|^2 - 2\int_{\Omega_1} M^\eta\cdot h_a e_1,$$

and we remark that for all $\eta>0$,

$$\tilde{\mathcal{E}}(M^\eta) \le \mathcal{E}^\eta(M^\eta)$$

so that for all $t\ge 0$,

$$\tilde{\mathcal{E}}(M^\eta(t)) + \int_0^t \left|\frac{\partial M^\eta}{\partial t}\right|^2 \le \tilde{\mathcal{E}}(m_0) + 2\int_0^t\int_{\Omega_1}|\partial_t h_a|.$$

By convexity arguments, taking the weak limit when η tends to zero, we obtain that for all $t\ge 0$,

$$A\int_{\Omega_1}\left|\frac{\partial M}{\partial x}\right|^2 + \int_{\mathbb{R}^3}|\mathcal{H}^0|^2 - 2\int_{\Omega_1} M\cdot h_a e_1 + \int_0^t\left|\frac{\partial M}{\partial t}\right|^2 \le \tilde{\mathcal{E}}(m_0) + 2\int_0^t\int_{\Omega_1}|\partial_t h_a|.$$

Let us describe now the limit for the demagnetizing field $\mathcal{H}^0$. For $i\in\{1,2,3\}$, we denote by M_i^η (resp $\mathcal{H}_i^\eta$) the coordinates of M^η (resp. $\mathcal{H}^\eta$), and we use the same notations for the limits M and $\mathcal{H}^0$. By rescaling equation (2.4) and by taking the weak limit, we obtain that:

$$\begin{cases} \partial_y(\mathcal{H}_2^0 + \overline{M_2}) + \partial_z(\mathcal{H}_3^0 + \overline{M_3}) = 0, \\ \partial_y\mathcal{H}_3^0 - \partial_z\mathcal{H}_2^0 = 0 \\ \partial_y\mathcal{H}_1^0 = 0, \ \ \partial_x\mathcal{H}_1^0 = 0. \end{cases}$$

Therefore since $\mathcal{H}^0\in L^2(\mathbb{R}^3)$, we obtain that $\mathcal{H}_1^0 = 0$ and that the transversal part of $\mathcal{H}^0$ is the 2-dimensional demagnetizing field calculated in the plane $\{x\}\times\mathbb{R}^2$ associated to the transversal part of M (that is $(M_2,M_3)(x,\cdot)$). Since the section of the wire is a ball of $\mathbb{R}^2$, since (M_2,M_3) is constant in the section of the wire, the calculation of the two-dimensional demagnetizing field generated by the constant configuration is classical (see [46] for the general result in the 3d case, or [20] for an elementary proof in 2d), and we obtain that $\mathcal{H}^0$ is given by:

$$\mathcal{H}^0(t,x,y,z) = \begin{cases} -\dfrac{1}{2}\begin{pmatrix} 0 \\ M_2(t,x) \\ M_3(t,x)\end{pmatrix} \text{ for } (x,y,z)\in\Omega_1 \\ \dfrac{1}{2}\dfrac{1}{(y^2+z^2)^2}\begin{pmatrix} 0 \\ M_2(t,x)(y^2-z^2) + 2M_3(t,x)yz \\ -M_3(t,x)(y^2-z^2) - 2M_2(t,x)yz\end{pmatrix} \text{ for } x\in[0,L] \text{ and } y^2+z^2>1 \\ 0 \text{ for } x\notin[0,L]. \end{cases}$$

Since for all $\eta>0$,

$$\int_{\mathbb{R}^3}|\mathcal{H}^\eta|^2 = -\int_{\Omega_1}\mathcal{H}^\eta\cdot M^\eta,$$

since $\mathcal{H}^\eta \rightharpoonup \mathcal{H}^0$ in L^2 weak $M^\eta \to M$ in L^2 strong, we obtain that

$$\int_{\mathbb{R}^3} |\mathcal{H}^\eta|^2 \to \frac{1}{2}\int_{\Omega_1} \left(|M_2|^2 + |M_3|^2\right) = \int_{\mathbb{R}^3} |\mathcal{H}^0|^2,$$

so that $\mathcal{H}^\eta \to \mathcal{H}^0$ strongly in $L^2(\mathbb{R}^3)$.

Remark 2.3. *In the three dimensional model, the demagnetizing field is a non local operator. We observe a localization of this operator in the asymptotic one-dimensional model. This localization is also observed in [49] for asymptotic studies in nano wires, and in [12] and [31] for two-dimensional models of ferromagnetic thin layers.*

Finally, we take in the weak formulation (2.12) test functions only depending on (t,x): $\Psi(t,x,y,z) = \psi(t,x)$, and taking the limit when η tends to zero we obtain that for all $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^+; H^1(0,L))$,

$$\begin{aligned}\int_{\mathbb{R}^+\times[0,L]} \left(\frac{\partial M}{\partial t} - M \times \frac{\partial M}{\partial t}\right)\cdot \psi = 2A\int_{\mathbb{R}^+\times[0,L]} \left(M \times \frac{\partial M}{\partial x}\right)\cdot \frac{\partial \psi}{\partial x} \\ +2\int_{\mathbb{R}^+\times[0,L]} M \times \left(\frac{1}{2}(M_2 e_2 + M_3 e_3) - h_a\, e_1\right)\cdot\psi.\end{aligned} \tag{2.13}$$

In conclusion, we obtain that $M : \mathbb{R}^+ \times [0,L] \to \mathbb{R}^3$ satisfies:

- $M \in L^\infty(\mathbb{R}^+; H^1(0,L))$ and $\dfrac{\partial M}{\partial t} \in L^2(\mathbb{R}^+ \times [0,L])$,
- $M(0,x) = m_0(x)$ in the trace sense (note that $M \in W^{1,2}(0,T; L^2([0,L]))$ so that $M \in \mathcal{C}^0([0,T]; L^2([0,L])))$.
- $|M(t,x)| = 1$ for almost every $(t,x) \in \mathbb{R}^+ \times [0,L]$,
- M satisfies (2.13),
- for all t,

$$\mathcal{E}(M(t)) + \int_0^t \int_{[0,L]} \left|\frac{\partial M}{\partial t}(s,x)\right|^2 dx\, ds \le \mathcal{E}(m_0) + 2\int_0^t \int_{[0,L]} |\partial_t h_a|, \tag{2.14}$$

with

$$\mathcal{E}(M(t)) = A\int_{[0,L]} \left|\frac{\partial M}{\partial x}(t,x)\right|^2 dx + \frac{1}{2}\int_{[0,L]} \left(|M_2|^2 + |M_3|^2\right)(t,x)dx - 2\int_{[0,L]} h_a(t,x)M_1(t,x)dx,$$

so that M is a weak solution of the one dimensional Landau-Lifschitz-Gilberg equation:

$$\frac{\partial M}{\partial t} - M \times \frac{\partial M}{\partial t} = -2M \times \left(A\partial_{xx}M - \frac{1}{2}(M_2 e_2 + M_3 e_3) + h_a(t,x)e_1\right),$$

with homogeneous Neumann Boundary conditions.

For regular solutions, this equation is equivalent to the following system we will study thereafter:

$$\begin{cases} \dfrac{\partial M}{\partial t} = -M \times H_e - M \times (M \times H_e), \\ H_e = A\partial_{xx}M - \dfrac{1}{2}(M_2 e_2 + M_3 e_3) + h_a e_1, \\ \partial_x M(t,0) = \partial_x M(t,L) = 0, \\ M(0,x) = m_0(x). \end{cases} \tag{2.15}$$

2.3 Model for ferromagnetic nanowires with electric current

In order to fit with the applications, we aim to obtain one dimensional models for a ferromagnetic wire submitted to an electric current. We start from the three-dimensional model of electric current described in [52], [53] and [54]. The electric current is modelled by an additional transport term of the form $(v \cdot \nabla)m + m \times \big((v \cdot \nabla))m\big)$, where $v(t,x)$ is a vector field directed along the direction of electrons motion, with an amplitude proportional to the current density (see [52]):

$$\begin{cases} \dfrac{\partial m}{\partial t} = -m \times H_e - m \times (m \times H_e) + (v \cdot \nabla)m + m \times \big((v \cdot \nabla))m\big), \\ H_e = A\Delta m + H_d(m), \\ \dfrac{\partial m}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.16}$$

Existence of weak solutions for (2.16) is tackled in [6]. For ferromagnetic nano wires with electric current, using the same asymptotic method as in the previous section, we obtain the following one-dimensional model:

$$\begin{cases} \dfrac{\partial M}{\partial t} = -M \times H_e(M) - M \times (M \times H_e(M)) + v\partial_x M + M \times v\partial_x M, \\ H_e(M) = A\partial_{xx}M - \dfrac{1}{2}(M_2 e_2 + M_3 e_3), \\ \partial_x M(t,0) = \partial_x M(t,L) = 0. \end{cases} \tag{2.17}$$

The parameter $v(t)$ is a scalar relied to the intensity of the applied current.

2.4 Domain Walls in Ferromagnetic Nanowires

A well known property of ferromagnetic materials is that the magnetization in a given sample tends to be structured in domains, large regions in which the magnetization is almost constant. The domains are separated by domain walls, thin zones with great variations of the magnetization distribution. Since the pioneering work of Walker (see [56]), there exists a huge literature in physics concerning the formation and the dynamics of the walls (see for example [47, 50, 52, 53, 54] and the references therein).

From the mathematical point of view, in the static case, the formation of walls for simplified models of two dimensional ferromagnetic devices is tackled in [1, 2, 25, 26, 48]. In the three-dimensional non static case, the interested reader should consult [13] and [29] for very partial results: the dynamics of domain walls in the 3d case remains essentially non-understood.

In the case of nanowires without applied field, the energy of a magnetization distribution $m : [0,L] \to S^2$ is given by

$$A \int_{[0,L]} |\partial_x m|^2 + \frac{1}{2} \int_{[0,L]} \left(|m_2|^2 + |m_3|^2\right).$$

Taking into account the saturation constraint $|m| = 1$, the minimization of the second part of the energy (coming from the demagnetizing energy) yields configurations taking the two values $-e_1$ and $+e_1$, but the presence of the exchange term does not allow discontinuities. The competition of these two terms induces the formation of large domains, in which m equals e_1 or $-e_1$, separated by domain walls of thickness $A^{\frac{1}{2}}$ as we will see after. This property is used to store digital information in nano wires, for instance by storing a bit 0 in a $-e_1$-domain, and a bit 1 in a $+e_1$-domain.

Our goal is to give a precise description of the walls and to explain the influence of an applied magnetic field or an electric current on the walls distribution. In the following section, we describe the walls with exact solutions of (2.15) and we study the stability of these exact solutions.

3 Exact Solutions Describing Domain Walls

3.1 Walls in infinite nanowires

In this section we deal with the following model of infinite nanowire with a constant applied field. The wire is assimilated to the real line $\mathbb{R}e_1$. The magnetization $m : \mathbb{R}_t^+ \times \mathbb{R}_x \to S^2$ satisfies the following system:

$$\begin{cases} \dfrac{\partial m}{\partial t} = -m \times (H_e(m) + h(t)e_1) - m \times (M \times (H_e(m) + h(t)e_1)), \\ \\ H_e(m) = \partial_{xx} m - (m_2 e_2 + m_3 e_3), \end{cases} \tag{3.1}$$

obtained from (2.15) by rescaling in the space variable $\tilde{x} = \frac{x}{\sqrt{2A}}$ and in the time variable $\tilde{t} = \frac{t}{2}$, and where $h(\tilde{t})$ is deduced from h_a by $h(\tilde{t}) = 2h_a(t)$ (the tilda variables are the new variables after rescaling, but we still denote them without tilda in the new model).

We remark that the system is invariant by translations in the space variable and by rotations around the wire axis, *i.e.* if m satisfies (3.1), then for $\sigma \in \mathbb{R}$ and $\theta \in \mathbb{R}$, the map $(t, x) \mapsto R_\theta m(t, x - \sigma)$ is solution for (3.1), with

$$R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}. \tag{3.2}$$

This invariance will play a crucial role for the obtention of the stability for solutions describing domain walls.

With a vanishing applied field $h = 0$, a domain wall separating a left hand side domain magnetized along $-e_1$ and a right hand side domain magnetized along $+e_1$ is described by the exact profile M^0 and all its translations-rotations, with:

$$M^0(x) = \begin{pmatrix} \tanh x \\ 1/\cosh x \\ 0 \end{pmatrix}. \tag{3.3}$$

Indeed, a straightforward calculation gives:

$$\partial_{xx} M^0 - (M_2^0 e_2 + M_3^0 e_3) = -\frac{2}{\cosh^2 x} M^0, \tag{3.4}$$

so $M^0 \times H_e(M^0) = 0$, that is M^0 is a stationary solution for equation (3.1) with $h = 0$.

For a non vanishing applied field $h(t)e_1$ (depending on time but constant along the wire for a given t), we can obtain exact solutions of (3.1) describing the dynamics of walls by the following way: let $\sigma_0^{ex} \in \mathbb{R}$ and $\theta_0^{ex} \in \mathbb{R}$ be given. We define $\sigma^{ex}(t)$ and $\theta^{ex}(t)$ by:

$$\begin{cases} \dfrac{d\sigma^{ex}}{dt}(t) = -h(t), \\ \\ \dfrac{d\theta^{ex}}{dt}(t) = h(t), \\ \\ \sigma^{ex}(0) = \sigma_0^{ex}, \quad \theta^{ex}(0) = \theta_0^{ex}. \end{cases} \tag{3.5}$$

Then,

$$\mathbf{m}^{ex} : (t, x) \mapsto R_{\theta^{ex}(t)} M^0(x - \sigma^{ex}(t)) \tag{3.6}$$

satisfies (3.1).

In [19], [21] and [35], the stability of such profiles and the effects of a non vanishing applied field on these configurations are studied. Roughly speaking, we have the following behavior.

Theorem 3.1. *Let $h \in \mathcal{C}^1(\mathbb{R}_t^+;\mathbb{R})$ satisfying*

$$|h(t)| \leq h_0 < 1 \text{ for all } t. \tag{3.7}$$

Let $(\theta_0^{ex}, \sigma_0^{ex}) \in \mathbb{R}^2$ and $(\theta^{ex}, \sigma^{ex})$ given by (3.5). We denote by $\mathbf{m}^{ex}$ the solution of (3.1) given by

$$\mathbf{m}^{ex}(t,x) = R_{\theta^{ex}(t)} M^0(x - \sigma^{ex}(t)).$$

Then for all $\varepsilon > 0$, there exists $\eta_0 > 0$ such that if $m_0 \in L^\infty(\mathbb{R}; S^2)$ satisfies

$$\|m_0 - \mathbf{m}^{ex}(0,\cdot)\|_{H^1(\mathbb{R})} \leq \eta,$$

then the solution m of (3.1) with initial data m_0 satisfies

$$\forall\, t \geq 0,\ \|m(t) - \mathbf{m}^{ex}(t,\cdot)\|_{H^1(\mathbb{R})} \leq \varepsilon \text{ (stability).}$$

In addition, there exists $(\theta_\infty, \sigma_\infty) \in \mathbb{R}^2$ such that

$$\|m(t,\cdot) - R_{\theta_\infty}\mathbf{m}^{ex}(t, \cdot - \sigma_\infty)\|_{H^1(\mathbb{R})} \underset{t\to+\infty}{\longrightarrow} 0 \text{ (asymptotic stability modulo rotation-translation).}$$

Remark 3.1. *This theorem contains a controlability result for the position of the wall, the control being the applied field (see [21]).*

Remark 3.2. *Assumption (3.7) on the applied field h is quite natural: for a constant applied field he_1 with $h \leq -1$ (resp. $h \geq 1$), then the constant solution describing only one domain given by $m = e_1$ (resp. $m = -e_1$) is unstable. Concerning the wall profiles, for a constant applied field he_1 with $|h| > 1$, the wall profile M^0 is linearly unstable for the Landau-Lifschitz equation (see [35]).*

The first difficulty of this problem is the saturation constraint: the perturbations which take values out of the sphere are irrelevant. So a perturbation cannot be written in the classical way by $\mathbf{m}^{ex} + w(t,x)$ where w is small, since it is not easy to check the saturation constraint under this form. The first idea is to describe the perturbation m is a convenient mobile frame, so that the saturation constraint is automatically satisfied.

The second difficulty is due to the invariance by rotations-translations : this induces that 0 is a double eigenvalue of the linearized equation. We use geometrical tools introduced for example in [34] to split the solution m into a part taking into account the rotations-translations of $\mathbf{m}^{ex}$, plus a part asymptotically decreasing to zero when $t \to +\infty$.

The third difficulty is that our problem is quasilinear, so that we must use variational methods to estimate the non linear terms.

Proof of theorem 3.1.

First step. Mobile Frame. In order to deal with a constant exact solution, we first perform the following change of unknown in (3.1): we denote by $u(t,x) = R_{-\theta(t)} m(t, x + \sigma(t))$ so that $m(t,x) = R_{\theta(t)} u(t, x - \sigma(t))$. We remark that $m = \mathbf{m}^{ex}$ is equivalent to $u = M^0$. In addition, m satisfies (3.1) if and only if u satisfies the following problem:

$$\frac{\partial u}{\partial t} = -u \times H_e(u) - u \times \big(u \times H_e(u)\big) - h\big(\partial_x u + u \times (u \times e_1)\big). \tag{3.8}$$

Furthermore, the stability of $\mathbf{m}^{ex}$ is equivalent to the stability of M^0 for Equation (3.8).

Now we aim to consider only perturbations u of M^0 satisfying the saturation constraint $|u| = 1$. We describe them in the mobile frame $\big(M^0(x), M^1(x), M^2\big)$ which vectors are defined by:

$$M^1(x) = \begin{pmatrix} -1/\cosh x \\ \tanh x \\ 0 \end{pmatrix} \text{ and } M^2 = M^0 \times M^1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{3.9}$$

writting:

$$u(t,x) = M^0(x) + r_1(t,x)M^1(x) + r_2(t,x)M^2 + \nu(r(t,x))M^0(x), \tag{3.10}$$

where $r = (r_1, r_2)$ will be the new unknown taking its values in a neighborhood of 0 in $\mathbb{R}^2$, and where $\nu : B_2(0,1/2) \to \mathbb{R}$ is given by

$$\nu(r_1, r_2) = \sqrt{1 - (r_1)^2 - (r_2)^2} - 1, \tag{3.11}$$

so that u satisfies automatically the saturation constraint $|u| = 1$.

Plugging (3.10) in (3.8) and taking the scalar product with M^1 and M^2, we obtain that u satisfies (3.8) if and only if r satisfies an equation of the form:

$$\frac{\partial r}{\partial t} = \Lambda r + h\ell r + F(x, h, r, \partial_x r, \partial_{xx} r), \tag{3.12}$$

where the linear part $\Lambda r + h\ell r$ is described by

$$\Lambda r = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} Lr_1 \\ Lr_2 \end{pmatrix}, \tag{3.13}$$

with

$$L = -\partial_{xx} + (1 - \frac{2}{\cosh^2 x}), \tag{3.14}$$

and

$$\ell = -\partial_x - \tanh x. \tag{3.15}$$

The non linear term $F(x, h, r, \partial_x r, \partial_{xx} r)$ is defined for r taking its values in $B_2(0,1/2)$. It has the following form:

$$F(x, h, r, \partial_x r, \partial_{xx} r) = G(r)\partial_{xx} r + H_1(x,r)(\frac{\partial r}{\partial x}) + H_2(r)(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial x}) + P(x, r, h), \tag{3.16}$$

with

- $G \in \mathcal{C}^\infty(B_2(0,1/2); \mathcal{M}^2(\mathbb{R}))$, where we denote by $\mathcal{M}^2(\mathbb{R})$ the set of the 2×2 real matrices. We have $G(r) = \mathcal{O}(|r|)$.
- $H_1 \in \mathcal{C}^\infty(\mathbb{R} \times B(0,1/2); \mathcal{M}^2(\mathbb{R}))$ and $H_1(x,r) = \mathcal{O}(|r|)$.
- $H_2 \in \mathcal{C}^\infty(B_2(0,1/2); \mathcal{L}_2(\mathbb{R}^2))$, where we denote by $\mathcal{L}_2(\mathbb{R}^2)$ the set of the bilinear applications defined on $\mathbb{R}^2 \times \mathbb{R}^2$ with values in $\mathbb{R}^2$. We have $H_2(x,r) = \mathcal{O}(|r|)$
- $P \in \mathcal{C}^\infty(\mathbb{R} \times B_2(0,1/2) \times \mathbb{R}; \mathbb{R}^2)$ with $P(x,r,h) = \mathcal{O}(|r|^2)$ uniformly in $x \in \mathbb{R}$ and for h in a bounded set.

Therefore, (3.12) is equivalent to (3.8). In addition, M^0 is stable for (3.8) if and only if 0 is stable for (3.12).

Third step: New Coordinates.
We remark that because of the invariance by rotations-translations for the Landau-Lifschitz equation (3.1), Equation (3.8) has the same property. For (θ, σ) in a neighborhood of zero in $\mathbb{R}^2$, we consider the coordinates of $R_\theta M^0(x - \sigma)$ in the mobile frame and we define $\mathcal{R}(\theta, \sigma)$ by:

$$\mathcal{R}(\theta, \sigma)(x) = \begin{pmatrix} M^1(x) \cdot R_\theta M^0(x - \sigma) \\ \\ M^2 \cdot R_\theta M^0(x - \sigma) \end{pmatrix}. \tag{3.17}$$

For all (θ, σ) in a neighborhood of zero, $\mathcal{R}(\theta, \sigma)$ is a solution for (3.12), so we have a two parameters family of static solutions. This induces that 0 is a double eigenvalue for the linear operator $\Lambda + h\ell$ associated to the linearized equation for (3.12). Indeed, the properties of Λ come from the properties of L summarized in the following proposition:

Proposition 3.1. *The linear operator L defined by (3.14) is a self-adjoint operator in $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R})$. It is positive. Its essential spectrum is $[1, +\infty[$. It admits zero as a simple eigenvalue, and zero is the unique eigenvalue of L.*

Proof. We remark that $L = \ell^* \circ \ell$, with $\ell = -\partial_x - \tanh x$, so that L is positive. The Kernel of L is obtained by solving $-\partial_x u - \tanh x\, u = 0$, and we have:

$$\text{Ker } L = \mathbb{R}\frac{1}{\cosh x}.$$

In addition, if v is an eigenvector for L, associated to the eigenvalue α, we have $Lv = \alpha v$, and by applying ℓ on this equality, since $\ell \circ \ell^* = -\partial_{xx} + 1$, we obtain that α is an eigenvalue for $-\partial_{xx} + 1$ associated to the eigenvector ℓv, which ensures that $\ell v = 0$ so that $\alpha = 0$.

We deduce from this proposition that 0 is a double eigenvalue for $\Lambda + h\ell$ associated to the eigenvectors $(\frac{1}{\cosh x}, 0)$ and $(0, \frac{1}{\cosh x})$. In addition, on the orthogonal of Ker L, we have the following property:

$$\text{if } < u|\frac{1}{\cosh x} >= 0, \;\; < Lu|u > \geq \|u\|^2_{L^2(\mathbb{R})}, \tag{3.18}$$

where we denote by $< \cdot|\cdot >$ the usual inner product in $L^2(\mathbb{R})$. Hence on $(\text{Ker } L)^\perp$, we can use the following norms equivalences:

$$\begin{aligned}
&\forall u \in H^2(\mathbb{R}) \cap (\text{Ker } L)^\perp, \;\; c_1\|u\|_{H^2(\mathbb{R})} \leq \|Lu\|_{L^2(\mathbb{R})} \leq c_2\|u\|_{H^2(\mathbb{R})}, \\
&\forall u \in H^1(\mathbb{R}) \cap (\text{Ker } L)^\perp, \;\; c_1\|u\|_{H^1(\mathbb{R})} \leq \|L^{\frac{1}{2}}u\|_{L^2(\mathbb{R})} \leq c_2\|u\|_{H^1(\mathbb{R})}.
\end{aligned} \tag{3.19}$$

Furthermore, we have

$$\forall u \in H^2(\mathbb{R}) \cap (\text{Ker } L)^\perp, \;\; \|\ell u\|_{L^2(\mathbb{R})} = \|L^{\frac{1}{2}}u\|_{L^2(\mathbb{R})} \leq \|Lu\|_{L^2(\mathbb{R})}. \tag{3.20}$$

The eigenvalue zero is always a difficulty to obtain the stability for a non linear problem. In order to take into account this problem, in a neighborhood of zero in $H^2(\mathbb{R};\mathbb{R}^2)$, we use a new parametrization writing:

$$r(x) = \mathcal{R}(\theta, \sigma)(x) + w(x),$$

with $(\theta, \sigma) \in \mathbb{R}^2$ and $w \in \mathcal{W} = \left(\text{Ker } L^\perp\right)^2$. Roughly speaking, for a fixed r, $\mathcal{R}(\theta, \sigma)$ is the projection parallel to $\mathcal{W}$ of r onto the surface of the exact solutions.

By using the local inversion theorem, the map $r \mapsto (\theta, \sigma, w)$ is a local diffeomorphism on a neighborhood of zero. So we use this parametrization to describe a perturbation r of zero, solution of equation (3.12) writing:

$$r(t, x) = \mathcal{R}(\theta(t), \sigma(t))(x) + w(t, x), \tag{3.21}$$

where the new unknowns are $(\theta, \sigma, w) \in \mathcal{C}^1(\mathbb{R}^+_t; \mathbb{R}^2 \times \mathcal{W})$. On $\mathcal{W}$, we will estimate w using the norm equivalence described in (3.19).

By plugging (3.21) in (3.12), by taking the projection of the obtained equation onto $(\text{Ker } L)^2$ and onto $\mathcal{W}$, we obtain an equivalent form for (3.12) written in the new unknowns:

$$\left\{\begin{aligned}
&\frac{\partial w}{\partial t} = \Lambda w + \ell w + \mathcal{K}(\sigma)w + \tilde{F}(x, \theta, \sigma, w, \partial_x w, \partial_{xx} w), \\
&\frac{d\theta}{dt} = K_1(\theta, \sigma, w), \\
&\frac{d\sigma}{dt} = K_2(\theta, \sigma, w),
\end{aligned}\right. \tag{3.22}$$

where

- the linear part $\Lambda w + \ell w$ for the first equation in the same as for (3.12) and is given by (3.13) and (3.15),

- the linear part $\mathcal{K}(\sigma)w$ is a perturbation satisfying:

$$\|\mathcal{K}(\sigma)w\|_{L^2(\mathbb{R})} \leq C_1|\sigma|\|w\|_{H^2(\mathbb{R})}, \tag{3.23}$$

- the non linear part for the first equation has the same form as the non linear part of (3.12) (see (3.16)), and we obtain that while $(\theta(t), \sigma(t), w(t))$ remains in a fixed neighborhood of zero,

$$\Big| < \tilde{F}(x, \theta, \sigma, w, \partial_x w, \partial_{xx} w)|Lw > \Big| \leq C_2 \|L^{\frac{1}{2}} w\|_{L^2(\mathbb{R})} \|Lw\|^2_{L^2(\mathbb{R})}, \tag{3.24}$$

- the right hand side terms K_1 and K_2 are obtained by projecting F onto Ker L. By integration by parts, they satisfy:

$$|K_i(\theta, \sigma, w)| \leq C_3 \|L^{\frac{1}{2}} w\|_{L^2(\mathbb{R})}, \tag{3.25}$$

  while (θ, σ, w) remains in a fixed neighborhood of zero.

In these new coordinates, using (3.19), Theorem 3.1 is equivalent to the following claim:

Claim. *Let $\varepsilon > 0$. There exists $\eta_0 > 0$ such that if $\|L^{\frac{1}{2}} w_0\|_{L^2(\mathbb{R})} + |\theta_0| + |\sigma_0| \leq \eta_0$, with $w_0 \in \mathcal{W}$, then the solution (θ, σ, w) of system (3.22) with initial data $(\theta_0, \sigma_0, w_0)$ satisfies:*

(i) for all $t > 0$, $\|L^{\frac{1}{2}} w(t)\|_{L^2(\mathbb{R})} + |\theta(t)| + |\sigma(t)| \leq \varepsilon$ (stability),

(ii) $\|L^{\frac{1}{2}} w(t)\|_{L^2(\mathbb{R})}$ tends to zero when t tends to $+\infty$ (asymptotic decreasing for the normal part),

(iii) there exists θ_∞ and σ_∞ such that $\theta(t) \to \theta_\infty$ and $\sigma(t) \to \sigma_\infty$ when $t \to +\infty$ (asymptotic stability modulo translations-rotations).

Forth step: proof of the claim.
Taking the $L^2(\mathbb{R})$-inner product of the first equation in (3.22) with Lw yields:

$$\frac{1}{2}\frac{d}{dt}\|L^{\frac{1}{2}} w\|^2_{L^2} + \|Lw\|^2_{L^2(\mathbb{R})} \leq h < \ell w|Lw > + < \mathcal{K}(\sigma)w|Lw > + < \tilde{F}(x, \theta, \sigma, w, \partial_x w, \partial_{xx} w)|Lw > .$$

From (3.20), we have

$$|< \ell w|Lw >| \leq \|Lw\|^2_{L^2(\mathbb{R})}.$$

In addition, the assumption on the applied field h ensures that $|h| \leq h_0 < 1$. Therefore, with (3.23) and (3.24), we get that while $\big(\theta(t), \sigma(t), w(t)\big)$ remains in a fixed neighborhood of zero, then

$$\frac{1}{2}\frac{d}{dt}\|L^{\frac{1}{2}} w\|^2_{L^2} + \|Lw\|^2_{L^2(\mathbb{R})} \leq (h_0 + C_1|\sigma|)\|Lw\|^2_{L^2(\mathbb{R})} + C_2\|Lw\|^2_{L^2(\mathbb{R})}\|L^{\frac{1}{2}} w\|_{L^2(\mathbb{R})},$$

therefore,

$$\frac{1}{2}\frac{d}{dt}\|L^{\frac{1}{2}} w\|^2_{L^2} + \|Lw\|^2_{L^2(\mathbb{R})}\Big(1 - h_0 - C_1|\sigma| - C_2\|L^{\frac{1}{2}} w\|_{L^2(\mathbb{R})}\Big) \leq 0. \tag{3.26}$$

Hence if

$$|\sigma| \leq \frac{1 - h_0}{2} \text{ and } \|L^{\frac{1}{2}} w_0\|_{L^2(\mathbb{R})} \leq \frac{1 - h_0}{4C_2}, \tag{3.27}$$

we obtain from (3.26) that $t \mapsto \|L^{\frac{1}{2}} w\|_{L^2(\mathbb{R})}$ is decreasing and thus remains smaller than $\frac{1 - h_0}{2C_1}$. So under assumption (3.27), while $(\theta(t), \sigma(t))$ remains in a fixed neighborhood of zero, inequality (3.26) remains valid and applying (3.20), we have:

$$\frac{1}{2}\frac{d}{dt}\|L^{\frac{1}{2}} w\|^2_{L^2} + \|L^{\frac{1}{2}} w\|^2_{L^2(\mathbb{R})}\frac{1 - h_0}{4} \leq 0,$$

so that: while $(\theta(t), \sigma(t))$ remains in a fixed neighborhood of zero, under assumption (3.27),

$$\|L^{\frac{1}{2}}w(t)\|^2_{L^2(\mathbb{R})} \leq \|L^{\frac{1}{2}}w_0\|^2_{L^2(\mathbb{R})}e^{-\frac{1-h_0}{4}t}. \tag{3.28}$$

Plugging the previous estimate in the equations on θ and σ in (3.22) together with Estimate (3.25) yield

$$\left|\frac{d\theta}{dt}(t)\right| \leq C_3\|L^{\frac{1}{2}}w_0\|_{L^2(\mathbb{R})}e^{-\frac{1-h_0}{4}t} \text{ and } \left|\frac{d\sigma}{dt}(t)\right| \leq C_3\|L^{\frac{1}{2}}w_0\|_{L^2(\mathbb{R})}e^{-\frac{1-h_0}{4}t}.$$

Therefore, integrating these inequalities, we obtain that if θ_0 and σ_0 are small, if w_0, satisfying Assumption (3.27), is sufficiently small, then on the one hand, $\theta(t)$ and $\sigma(t)$ remain in the fixed neighborhood of zero so that the previous estimates remains valid for all time and on the other hand, $\frac{d\theta}{dt}$ and $\frac{d\sigma}{dt}$ are integrable on $\mathbb{R}$ so $\theta(t)$ (resp. $\sigma(t)$) admits a limit θ_∞ (resp. σ_∞) when t tends to $+\infty$.
From (3.28), under the previous assumptions, $w(t)$ tends to 0 when t tends to $+\infty$. This concludes the proof of Theorem 3.1.

3.2 Electric current in ferromagnetic nanonwires

The stability for profiles describing the wall motion induced by an electric current is tackled in [36]. This result is important from the point of view of the physics since for the most part of the applications (for example in racetrack memories), an electric current is used for walls motion. The main advantages of this solution compared to the applied magnetic field are the following: on the one hand it is easier to generate a constant electric field in a wire, even if it is not straight. On the other hand, a constant applied current induces a motion of the walls preserving their positions one with respect to each other whereas a constant applied magnetic field in a finite wire can induce the collapse of consecutive walls and so the annihilation of domains. We recall that this applied current is modelled by additional transport terms in the Landau-Lifschitz equation, so that, after rescaling, we deal with the following model:

$$\begin{cases} m : \mathbb{R}^+_t \times \mathbb{R}_x \to S^2, \\ \dfrac{\partial m}{\partial t} = -m \times H_e(m) - m \times (M \times H_e(m)) + v\partial_x m + m \times v\partial_x m, \\ H_e(m) = \partial_{xx}m - (m_2e_2 + m_3e_3). \end{cases} \tag{3.29}$$

For a constant applied courant v, a solution of (3.29) is given by $\mathbf{m}^v(t,x) = R_{-vt}M^0(x+vt)$. Using the same method as in the previous part, the stability of $\mathbf{m}^v$ is proved for $|v| < 2$ (see [36]). After writing an equivalent formulation is a convenient mobile frame, after splitting the new unknown in a part taking into account the invariance by translation-rotation plus a part w taking its values in $\mathcal{W} = (\text{Ker } L)^\perp \times (\text{Ker } L)^\perp$, the key point is to obtain the coercivity for the linear operator Λ_v given by:

$$\Lambda_v w = J\begin{pmatrix} Lw_1 \\ Lw_2 \end{pmatrix} + v\begin{pmatrix} \ell w_2 \\ -\ell w_1 \end{pmatrix}, \tag{3.30}$$

with $\mathcal{D}(\Lambda_v) = \mathcal{W} \cap H^2(\mathbb{R})$. Taking the inner product of $\Lambda_v(w)$ with Lw, we obtain:

$$< \Lambda_v(w)|Lw >= -\|Lw\|^2_{L^2(\mathbb{R})} + v\big(< \ell w_2|Lw_1 > - < \ell w_1|Lw_2 > \big).$$

On the one hand,

$$\begin{aligned} \|Lw_i\|^2_{L^2(\mathbb{R})} &= \ < \ell^*\ell w_i|\ell^*\ell w_i > \\ &= \ < \ell\ell^*\ell w_i|\ell w_i > \\ &= \ < (1+|\xi|^2)\mathcal{F}(\ell w_i)|\mathcal{F}(\ell w_i) > \\ &= \ \|\sqrt{1+|\xi|^2}\mathcal{F}(\ell w_i)\|^2_{L^2(\mathbb{R})}, \end{aligned}$$

where $\mathcal{F}$ is the Fourier transform, and using that $\ell\ell^* = -\partial_{xx} + 1$.
On the other hand,

$$
\begin{aligned}
|< \ell w_2 | L w_1 > - < \ell w_1 | L w_2 >| = & \; 2 \left| \int_{\mathbb{R}} \ell w_2 \partial_x (\ell w_1) \right| \\
\leq & \; 2 \, |< \mathcal{F}(\ell w_2) | \mathrm{i}\xi \mathcal{F}(\ell w_1) >| \\
\leq & \int_{\mathbb{R}} (1 + |\xi|^2) |\mathcal{F}(\ell w_1)| \, |\mathcal{F}(\ell w_2)| \\
\leq & \; \| \sqrt{1 + |\xi|^2} \mathcal{F}(\ell w_1) \|_{L^2(\mathbb{R})} \| \sqrt{1 + |\xi|^2} \mathcal{F}(\ell w_2) \|_{L^2(\mathbb{R})} \\
\leq & \; \frac{1}{2} \left(\| L w_1 \|^2_{L^2(\mathbb{R})} + \| L w_2 \|^2_{L^2(\mathbb{R})} \right).
\end{aligned}
$$

So for $|v| < 2$ we can compensate the perturbation term due to the applied current by the main term $\|Lw\|^2_{L^2(\mathbb{R})}$ so that we obtain the coercivity for the linearized operator Λ_v.

3.3 Walls in finite nanowires

In [20] we study the existence and stability of a one wall configuration for a one-dimensional model of finite wire. After rescaling, the model is the following: the magnetic moment m is defined on $\mathbb{R}_t^+ \times [0, L/\sqrt{2A}]$ with values in S^2 and satisfies the following Landau-Lifschitz equation:

$$
\begin{cases}
\dfrac{\partial m}{\partial t} = -m \times (H_e(m) + h(t)e_1) - m \times (M \times (H_e(m) + h(t)e_1)), \\
H_e(m) = \partial_{xx} m - (m_2 e_2 + m_3 e_3), \\
\partial_x m(t, 0) = \partial_x m(t, L/\sqrt{2A}) = 0 \text{ (Neumann homogeneous boundary conditions).}
\end{cases}
\tag{3.31}
$$

For a vanishing applied field, we look for a static solution describing one wall of the form:

$$
M_0(x) = \begin{pmatrix} \sin\theta_0 \\ \cos\theta_0 \\ 0 \end{pmatrix}.
$$

We find that M_0 is a static solution of (3.31) if and only if θ_0 satisfies the pendulum equation with homogeneous Neumann boundary conditions:

$$
\begin{cases}
-\theta_0'' - \sin\theta_0 \cos\theta_0 = 0 \text{ on } [0, L/\sqrt{2A}], \\
\theta_0'(0) = \theta_0'(L/\sqrt{2A}) = 0.
\end{cases}
\tag{3.32}
$$

We look for solutions describing only one wall, so we only consider solutions satisfying $-\pi/2 \leq \theta(0) < 0 < \theta(L/\sqrt{2A}) \leq \pi/2$. Our first result is that this solution exists if and only if $\dfrac{L}{\sqrt{2A}} > \dfrac{\pi}{2}$, *i.e.* the wire has to be long enough to contain a wall. We should develop the same method as for an infinite wire to study the stability of this static profile. We write the perturbations m of M_0 in a mobile frame following the studied profile as:

$$
m(t,x) = M_0(x) + r_1(t,x) \begin{pmatrix} -\cos\theta_0 \\ \sin\theta_0 \\ 0 \end{pmatrix} + r_2(t,x) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \nu(r(t,x)) M_0(x),
$$

where ν is defined by (3.11) and we obtain for r an equivalent equation of the form:

$$
\frac{\partial r}{\partial t} = \widetilde{\Lambda} r + F(x, r, \partial_x r, \partial_{xx} r),
\tag{3.33}
$$

where F is the non linear part, and where the linear term $\widetilde{\Lambda}r$ writes:

$$\widetilde{\Lambda}r = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} (\tilde{L} - \cos^2\gamma_0)r_1 \\ \tilde{L}r_2 \end{pmatrix}, \tag{3.34}$$

with

$$\tilde{L} = -\partial_{xx} + \sin^2\theta_0 - (\theta_0')^2, \text{ and } \gamma_0 = \theta_0(0). \tag{3.35}$$

As in the infinite case, we prove that $\tilde{L}$ is self-adjoint and positive since it can be factorized as $\tilde{L} = \tilde{\ell}^* \circ \tilde{\ell}$, with $\tilde{\ell} = \partial_x + \theta_0' \tan\theta_0$. We remark that Ker $\tilde{\mathcal{L}} = \mathbb{R}\cos\theta_0$ and that the second eigenvalue of $\tilde{\mathcal{L}}$ is 1, since $\sin\theta_0$ vanishes once in the domain and satisfies $\tilde{L}(\sin\theta_0) = \sin\theta_0$.
We have then

$$< \tilde{\Lambda} \begin{pmatrix} \cos\theta_0 \\ 0 \end{pmatrix} | \begin{pmatrix} \cos\theta_0 \\ 0 \end{pmatrix} >= \int_{[0,L/\sqrt{2A}]} \cos^2\gamma_0 \cos^2\theta_0(x)dx > 0,$$

which implies that the solution 0 is linearly unstable for (3.33). Therefore, in the case of finite wires, the exact solution describing one wall is linearly unstable for the Landau-Lifschitz equation (3.31). We remark that the eigenvector $\begin{pmatrix} \cos\theta_0 \\ 0 \end{pmatrix}$ is linked with the translations of the wall profile. Roughly speaking, the Landau-Lifschitz equation on finite wires can decrease the energy of the wall by translating it and finally by pushing it outside the wire, so that the one wall configuration is unstable.

Nevertheless, we prove in [20] that it is possible to stabilize the wall profile with an adapted magnetic field, but this is irrelevant from the point of view of the applications since we aim to obtain ferromagnetic devices storing the digital information without injecting energy in the system.

In [42], the authors study distributions of several walls in a periodic nanowire modelling ferromagnetic rings. They look for $L/\sqrt{2A}$-periodic solutions of the one dimensional model (3.1), where L is the length of the ring. They describe all these solutions and they prove that they are unstable.

3.4 Conclusion for the exacts solutions

In order to describe walls distributions in a finite nanowire, the exact solutions are inappropriate since they are unstable. In addition, it is impossible to describe with these solutions a configuration with several walls located at arbitrary places, since the exact solutions can only describe periodic positions for the walls (they are obtained by solving a pendulum equation which solutions are periodic).
Therefore, in order to describe realistic patterns of several walls located at arbitrary positions in finite nanowires, we have to deal with approximate solutions, and to prove that these quasi-solutions are metastable as we will see in the following part.

4 Quasi-Solutions

In this Section, we deal with the following model of finite nanowire:

$$\begin{cases} \partial_t m = -m \times \left(h^\varepsilon(m) + \frac{1}{\varepsilon}he_1\right) - m \times \left(m \times \left(h^\varepsilon(m) + \frac{1}{\varepsilon}he_1\right)\right)), \\ h^\varepsilon(m) = \varepsilon\partial_{xx}m - \frac{1}{\varepsilon}\left(m_2e_2 + m_3e_3\right), \\ \partial_x m(0) = \partial_x m(L/\sqrt{2}) = 0. \end{cases} \tag{4.1}$$

This model is obtained from (2.15) by writing $A = \varepsilon^2$, by rescaling in x ($\tilde{x} = x/\sqrt{2}$), by writing $h = 2h_a$, and by rescaling in time $\tilde{t} = \frac{\varepsilon}{2}t$, so that we describe the long time behavior of the solutions.

We remark that this rescaling in time induces the presence of stiff terms in the effective field. We aim to describe with this model the evolution of N walls (where N is arbitrary fixed) separating $N+1$ domains (magnetized along $-e_1$ or $+e_1$), when we apply a magnetic field $h(t,x)e_1$. Our analysis is based on the fact that for physical applications, the exchange length is small compared to the length of the wire, that is our exchange coefficient ε^2 is small.

Our approach is inspired by the famous paper of Carr and Pego [22]. They study the metastability for quasi-solutions of the Allen-Cahn model of phase transitions:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon^2 \partial_{xx} u - f(u), \\ \partial_x u(0,t) = \partial_x u(1,t), \\ u : \mathbb{R}_t^+ \times [0,1]_x \to \mathbb{R}, \end{cases} \tag{4.2}$$

where $f = F'$ is derived from a two wells potential F with two non degenerate minima at the points -1 and +1 (for example, $F(u) = (u^2-1)^2$). For small ε, they construct a N-parameters family $\mathcal{M}$ of quasi-solutions u^h describing N phase transitions located at the positions $h = (h_1, h_2, \dots, h_N)$, and they prove that these quasi-solutions are persistent on a time scale of order $\mathcal{O}(e^{\frac{c}{\varepsilon}})$ (see also [28] for related results). In a neighborhood of the manifold $\mathcal{M}$, the solution u of (4.2) is described as:

$$u(t,x) = u^{h(t)}(x) + v(t,x), \tag{4.3}$$

where $u^{h(t)}$ is the orthogonal projection of $u(t,\cdot)$ onto $\mathcal{M}$, so that $v(t,\cdot) \in (T_{u^h}\mathcal{M})^\perp$. Because of the spectral properties of the linearized equation for v, they show that v is exponentially decreasing so that $u(t)$ remains for all time very close to $u^{h(t)}$, so that the dynamics of u is essentially described by the very slow dynamics of the phase transitions.

The same method for the Landau-Lifschitz model (4.1) entails new technical difficulties. Following the same strategy, we construct a family of quasi-solutions describing the distributions of N walls. We are here in a vectorial case, so that our family is $2N$-dimensional (taking into account the positions of the walls, and the "tilts" of the profiles). Concerning the new coordinates close to the manifold of quasi-solutions, analogous to those used by Carr and Pego in (4.3), we have now to take into account the saturation constraint satisfied by the magnetic moment: $|m| = 1$. The estimates for the non linear terms are more difficult in our case since the problem is quasilinear (because of the non linear precession term $m \times \partial_{xx} m$). Furthermore, in our case, we are able to describe the motion of walls induced by the applied magnetic field h.

Persistence of phase transitions patterns for the Allen-Cahn equation is also obtained by Bronsard and Kohn in [9] with energetic considerations. See also [5] and [45] for the same kind of problem in a vectorial framework.

4.1 Construction of approximate solutions

We first construct configurations of N walls with a vanishing applied field. We only deal with configurations in which the walls are not too close one to each other and are quite far from the ends of the wire. We fix a lower bound $\delta > 0$, with $N\delta \ll L$. The walls are supposed to be located at the points $\sigma_1, \dots, \sigma_N$, satisfying:

$$\begin{cases} 0 < \sigma_1 - \delta, \\ \sigma_i + \delta < \sigma_{i+1} - \delta \text{ for } i \in \{1, \dots, N-1\}, \\ \sigma_N + \delta < L, \end{cases} \tag{4.4}$$

that is the distance between two consecutive walls is greater than 2δ, and the distance between a wall and the boundary is greater than δ. We denote Σ_δ the set of the $(\sigma_1, \dots, \sigma_N) \in \mathbb{R}^N$ satisfying (4.4).

For $\sigma \in \Sigma_\delta$ and for $\theta = (\theta_1, \dots, \theta_N) \in \mathbb{R}^N$, we construct the profile $\mathbf{m}_\varepsilon(\theta, \sigma)$ in the following way. Roughly speaking, in the domains, the magnetization equals $-e_1$ or $+e_1$. In a wall, we distinguish a central zone in which the magnetization is described by rescaling the exact solution M^0 given by (3.3). This central zone in surrounded by two transitional zones connecting smoothly the profiles in the domain on one hand and in the central zone on the other hand. In order to define precisely these profiles, we introduce a cut off function $\psi : \mathbb{R} \to [0, 1]$, such that $\psi(s) = 0$ for $s \leq \dfrac{3\delta}{4}$ and $\psi(s) = 1$ for $s \geq \dfrac{7\delta}{8}$.

Concerning the domains:

- on the first left hand side domain $[0, \sigma_1 - \delta]$, $\mathbf{m}_\varepsilon(\theta, \sigma)(x) = -e_1$,
- on the domain $[\sigma_i + \delta, \sigma_{i+1} - \delta]$, $i \in \{1, \dots, N-1\}$, $\mathbf{m}_\varepsilon(\theta, \sigma)(x) = (-1)^{i+1} e_1$,
- on the last right hand side domain $[\sigma_N + \delta, L]$, $\mathbf{m}_\varepsilon(\theta, \sigma)(x) = (-1)^{N+1} e_1$.

Concerning the wall i, connecting a $(-1)^i e_1$ left hand side domain to a $(-1)^{i+1} e_1$ right hand side domain, the key point is that $z \mapsto (-1)^{i+1} M^0(\dfrac{z}{\varepsilon})$ is an exact solution describing such a wall for (4.1) with vanishing applied field in an infinite wire.

The profile $\mathbf{m}_\varepsilon(\theta, \sigma)$ is defined as follows in the wall zone $[\sigma_i - \delta, \sigma_i + \delta]$: we remark that M^0 defined by (3.3) satisfies:

$$M^0(z) = \begin{pmatrix} \sin \arcsin \tanh z \\ \\ \cos \arcsin \tanh z \\ \\ 0 \end{pmatrix}.$$

We define $\varphi_\varepsilon^\delta : [-\delta, \delta] \to \mathbb{R}$ by

$$\varphi_\varepsilon^\delta(z) = \begin{cases} \arcsin \tanh \dfrac{z}{\varepsilon} & \text{for } -\delta/2 \leq z \leq \delta/2 \ \text{(central zone)}, \\ \\ -\dfrac{\pi}{2}\psi(-z) + (1 - \psi(-z)) \arcsin \tanh \dfrac{z}{\varepsilon} & \text{for } -\delta \leq z \leq -\delta/2 \ \text{(left transitional zone)}, \\ \\ \dfrac{\pi}{2}\psi(z) + (1 - \psi(z)) \arcsin \tanh \dfrac{z}{\varepsilon} & \text{for } \delta/2 \leq z \leq \delta \ \text{(right transitional zone)}, \end{cases}$$

so that $\varphi_\varepsilon^\delta$ equals $\arcsin \tanh \dfrac{z}{\varepsilon}$ in a central zone $[-\delta/2, \delta/2]$ and connects smoothly this profile to $-\dfrac{\pi}{2}$ at the left hand side and to $+\dfrac{\pi}{2}$ at the right hand side. Then we define $\mathbf{m}_\varepsilon(\theta, \sigma)$ in the wall $[\sigma_i - \delta, \sigma_i + \delta]$ by

$$\mathbf{m}_\varepsilon(\theta, \sigma)(x) = (-1)^{i+1} R_{\frac{\theta_i}{\varepsilon}} \begin{pmatrix} \sin \varphi_\varepsilon^\delta(x - \sigma_i) \\ \\ \cos \varphi_\varepsilon^\delta(x - \sigma_i) \\ \\ 0 \end{pmatrix}. \tag{4.5}$$

The profile defined above satisfies (4.1) with vanishing applied field excepted in the transitional zones, in which it is very close to $-e_1$ or $+e_1$ when ε is small.

When the applied field is non vanishing, the dynamics for walls is described using our quasi-solutions. We assume that the applied field h satisfies:

$$\begin{cases} h \in \mathcal{C}^2(\mathbb{R}^+ \times [0, L]; \mathbb{R}), \\ \\ \forall\, (t, x),\ |h(t, x)| \leq h_0 < 1, \\ \\ \exists\, C,\ \forall\, (t, x),\ |\partial_x h(t, x)| + |\partial_{xx} h(t, x)| \leq C. \end{cases} \tag{4.6}$$

For an initial set of positions $\overline{\sigma} \in \Sigma_\delta$ and an initial set of angles $\overline{\theta} \in \mathbb{R}^N$, we consider $(\theta^{ref}, \sigma^{ref}) \in \mathcal{C}^1(\mathbb{R}^+; \mathbb{R}^N \times \mathbb{R}^N)$ the solution of

$$\begin{cases} \dfrac{d\sigma_i^{ref}}{dt} = (-1)^i h(t, \sigma_i^{ref}), \\ \\ \dfrac{d\theta_i^{ref}}{dt} = h(t, \sigma_i^{ref}), \\ \\ \sigma^{ref}(t=0) = \overline{\sigma},\ \theta^{ref}(t=0) = \overline{\theta}. \end{cases} \tag{4.7}$$

While $\sigma^{ref}(t)$ remains in Σ_δ, the dynamics of walls is described by the profile:

$$(t, x) \mapsto \mathbf{m}_\varepsilon(\theta^{ref}(t), \sigma^{ref}(t))(x),$$

i.e. the above profile is almost solution for (4.1) with a non vanishing applied field.
We aim to prove that the exact solution with initial data close to $\mathbf{m}_\varepsilon(\overline{\theta}, \overline{\sigma})$ remains close to the above profile in a large time interval. The key point of our analysis is to rewrite equation (4.1) is new coordinates while m remains close to the set of quasi-solutions.

We denote by $\mathcal{M}_\delta$ the set:

$$\mathcal{M}_\delta = \left\{ \mathbf{m}_\varepsilon(\theta, \sigma), \theta \in \mathbb{R}^N; \sigma \in \Sigma_\delta \right\}.$$

This set is a 2N-dimensional submanifold of $H^1([0, L]; S^2)$, its boundary corresponds to the case when two walls are too close to each other, or when a wall is too close to one end of the wire. We parametrize a neighborhood of $\mathcal{M}_\delta$ by:

$$m = \mathbf{m}_\varepsilon(\theta, \sigma) + w + \nu(w)\mathbf{m}_\varepsilon(\theta, \sigma), \tag{4.8}$$

where

- $\theta \in \mathbb{R}^N$,
- $\sigma \in \Sigma_\delta$,
- ν is defined in (3.11),
- $w \in \mathcal{W}^\varepsilon_{\theta,\sigma}$, where $\mathcal{W}^\varepsilon_{\theta,\sigma}$ is analogous to the normal space to $\mathcal{M}_\delta$ at the point $\mathbf{m}_\varepsilon(\theta, \sigma)$: it is the set of the $w \in H^1([0, L]; \mathbb{R}^3)$ satisfying

$$\begin{aligned} &(i) \quad \forall\, x \in [0, L], w(x) \cdot \mathbf{m}_\varepsilon(\theta, \sigma)(x) = 0, \\ &(ii) \quad \forall\, i \in \{1, \dots, N\}, < \partial_{\sigma_i} \mathbf{m}_\varepsilon(\theta, \sigma) | w >= 0 \\ &(iii) \quad \forall\, i \in \{1, \dots, N\}, < \partial_{\theta_i} \mathbf{m}_\varepsilon(\theta, \sigma) | w >= 0. \end{aligned} \tag{4.9}$$

Property (i) together with the definition of ν ensure that m given by (4.8) satisfies the constraint $|m| = 1$. Orthogonality conditions (ii) and (iii) ensure that w takes its values in the normal bundle of the manifold $\mathcal{M}_\delta$.

Using the local inversion theorem, we can prove that this system of coordinates remains valid in a neighborhood of $\mathcal{M}_\delta$ which size (for the L^∞ norm) is independent of ε. We will work now with these new coordinates. We endow $\mathcal{W}^\varepsilon_{\theta,\sigma}$ with the norm:

$$\|w\|_\varepsilon = \left(\varepsilon \|\partial_x w\|^2_{L^2([0,L])} + \frac{1}{\varepsilon} \|w\|^2_{L^2([0,L])} \right)^{\frac{1}{2}}.$$

We establish in [14] the following result.

Theorem 4.1. *Let h, θ^{ref} and σ^{ref} satisfying (4.6) and (4.7). We assume that for all t, $\sigma^{ref}(t) \in \Sigma_{2\delta}$ and that:*

$$\forall\, t \geq 0, \quad \forall\, i \in \{1, \dots, N\}, \quad \forall\, x \in [\sigma_i^{ref}(t) - 2\delta, \sigma_i^{ref}(t) + 2\delta], \quad \partial_x h(t, x) = 0. \tag{4.10}$$

For $\nu_0 > 0$, there exists $\alpha_0 > 0$, there exists K such that for all $\varepsilon > 0$ we have:
for all $\sigma_0 \in \Sigma_{2\delta}$ with $|\sigma_0 - \overline{\sigma}| \leq \alpha_0$, for all $\theta_0 \in \mathbb{R}^N$ such that $|\theta_0 - \overline{\theta}| \leq \alpha_0$, for all $w_0 \in \mathcal{W}^{\varepsilon}_{\theta_0, \sigma_0}$ such that $\|w_0\|_\varepsilon \leq \alpha_0$, the solution m of (4.1) with initial data $m_0 = \mathbf{m}_\varepsilon(\theta_0, \sigma_0) + w_0 + \nu(w_0)\mathbf{m}_\varepsilon(\theta_0, \sigma_0)$ can be written as

$$m(t) = \mathbf{m}_\varepsilon(\theta(t), \sigma(t)) + w(t) + \nu(w(t))\mathbf{m}_\varepsilon(\theta(t), \sigma(t)),$$

with, for all $t \in [0, K e^{\frac{\delta}{4\varepsilon}}]$,

- *$\sigma(t) \in \Sigma_\delta$ and $|\sigma(t) - \sigma^{ref}(t)| \leq \nu_0$,*
- *$|\theta(t) - \theta^{ref}(t)| \leq \nu_0$,*
- *$\|w(t)\|_\varepsilon \leq \nu_0$.*

This theorem establishes that the dynamics of the solutions of (4.1) is essentially described by the approximate solution $\mathbf{m}_\varepsilon(\theta^{ref}, \sigma^{ref})$ in an exponentially large time interval. In fact, one can prove that the solution of (4.1) remains exponentially close to $\mathcal{M}_\delta$ while it does not arrive at the boundary of this manifold.
If we relax Assumption (4.10) then we obtain the same control of the solution, but on a shorter time interval of size $\mathcal{O}(\frac{1}{\varepsilon})$. This is due to the fact that $\mathbf{m}_\varepsilon(\theta^{ref}(t), \sigma^{ref}(t))$ is not a so good approximate solution without this assumption (the error is of order $\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$ with (4.10) and of order $\mathcal{O}(\varepsilon)$ for a non constant applied field in the wall).

4.2 Sketch of the proof of Theorem 4.1

First step: Landau-Lifschitz equation in the new coordinates.

We plug (4.8) in (4.1), and by taking the L^2 inner product with $\partial_{\theta_i}\mathbf{m}_\varepsilon$ and $\partial_{\sigma_i}\mathbf{m}_\varepsilon$, by using the orthogonality conditions (ii) and (iii) in (4.9), we obtain the following system for $(\theta(t), \sigma(t))$:

$$\begin{cases} \dfrac{d\theta_i}{dt} = h_i + a_\varepsilon^1 + G_\varepsilon^1(\theta_i, \sigma_i, w), \\[2ex] \dfrac{d\sigma_i}{dt} = (-1)^i h_i + a_\varepsilon^2 + G_\varepsilon^2(\theta_i, \sigma_i, w), \end{cases} \tag{4.11}$$

where

- $h_i(t)$ is the mean value of $h(t, \cdot)$ in the central zone $[\sigma_i - \delta/2, \sigma_i + \delta/2]$ for the i^{th} wall.
- The corrector terms a_ε^i come from the fact that the profile $\mathbf{m}_\varepsilon(\theta, \sigma)$ is only an approximate solution for (4.1). In particular, under Assumption (4.10), $a_\varepsilon^i = \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$ and without this assumption, $a_\varepsilon^i = \mathcal{O}(\varepsilon^2)$.
- The terms G_ε^i are estimated as follows: while σ remains in Σ_δ,

$$\left| G_\varepsilon^i(\theta_i, \sigma_i, w) \right| \leq C \|w\|_\varepsilon. \tag{4.12}$$

We remark that if w is small, then Equation (4.11) is a small perturbation of the equation (4.7) satisfied by the reference profile $\mathbf{m}_\varepsilon(\theta^{ref}, \sigma^{ref})$.

The equation for the normal part w is of the form:

$$\frac{\partial w}{\partial t} = a_\varepsilon + \Lambda_\varepsilon + P_\varepsilon w + l_\varepsilon w + \mathcal{G}^\varepsilon(w, \theta, \sigma), \tag{4.13}$$

where

- a_ε is a corrector term of order $\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$,
- l_ε is a corrector term for the linear part, with $\|l_\varepsilon w\|_{L^2} \leq \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})\|w\|_\varepsilon$,
- the linear term Λ_ε is defined by:

$$\Lambda_\varepsilon w = -\mathbf{m}_\varepsilon \times L^\varepsilon(w) - \mathbf{m}_\varepsilon \times (\mathbf{m}_\varepsilon \times L^\varepsilon(w)), \text{ with } L^\varepsilon(w) = -\varepsilon\partial_{xx}w - \frac{1}{\varepsilon}w_1 e_1 + f_\varepsilon^\sigma w, \quad (4.14)$$

where

$$f_\varepsilon^\sigma(x) = \begin{cases} \frac{1}{\varepsilon} \text{ for } x \text{ in the domains,} \\ \frac{1}{\varepsilon}\left(1 - \frac{2}{\cosh^2(\frac{x-\sigma_i}{\varepsilon})}\right) \text{ for } x \text{ in the central zone of the } i^{th} \text{ wall,} \\ \frac{1}{\varepsilon} + \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}) \text{ in the transitional zones of the walls.} \end{cases}$$

- P_ε is the linear part due to the applied magnetic field h.
- The non linear part $\mathcal{G}^\varepsilon$ satisfies:

$$\|\mathcal{G}^\varepsilon(w,\theta,\sigma)\|_{L^2} \leq K\|w\|_\varepsilon \left(\|\varepsilon\partial_{xx}w\|_{L^2} + \frac{1}{\varepsilon}\|w\|_{L^2}\right). \quad (4.15)$$

The previous estimates obtained with assumption (4.10) are valid while $\sigma(t)$ remains in Σ_δ. They are weakened without assumption (4.10) since in this case, the corrector terms a_ε and l_ε are of order $\mathcal{O}(\sqrt{\varepsilon})$.

Second Step: coercivity for the operator L^ε.

We aim to prove that the operator L^ε (which plays the same role as L in Section 3) satisfies a coercivity condition of the form:

$$\forall\, w \in \mathcal{W}_{\theta,\sigma}^\varepsilon,\ < L^\varepsilon(w)|w > \geq \frac{1}{\varepsilon}\|w\|_{L^2}^2.$$

If w has its support in a domain, the previous estimate is clear: in this case, since $\mathbf{m}_\varepsilon(\theta,\sigma) = \pm e_1$ and since $w \cdot \mathbf{m}_\varepsilon(\theta,\sigma) = 0$ (point wise orthogonality condition), then $w_1 = 0$. So

$$< L^\varepsilon(w)|w > = < -\varepsilon\partial_{xx}w + \frac{1}{\varepsilon}w|w > = \varepsilon\|\partial_x w\|_{L^2}^2 + \frac{1}{\varepsilon}\|w\|_{L^2}^2$$

by integration by parts.

If w has its support in the wall $[\sigma_i - \delta, \sigma_i + \delta]$, we describe w in a mobile frame inspired from the one used in Section 3. Writing w on the form:

$$w(t,w) = r_1(\frac{x-\sigma_i}{\varepsilon})R_{\frac{\theta_i}{\varepsilon}}M_1(\frac{x-\sigma_i}{\varepsilon}) + r_2(\frac{x-\sigma_i}{\varepsilon})R_{\frac{\theta_i}{\varepsilon}}M_2, \quad (4.16)$$

where M_1 and M_2 are defined in (3.9) in Section 3, then the point wise orthogonality condition (i) in (4.9) is automatically satisfied. In these new coordinates, we obtain that

$$< L^\varepsilon(w)|w > = < Lr_1|r_1 > + < Lr_2|r_2 >,$$

where L is the linear operator appearing in Section 3 defined by (3.14). The orthogonality conditions (ii) and (iii) in (4.9) imply a quasi-orthogonality condition for r_1 and r_2, that is we obtain that:

$$< r_1|\frac{1}{\cosh x} > = \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})\|w\|_{L^2} \text{ and } < r_2|\frac{1}{\cosh x} > = \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})\|w\|_{L^2},$$

so that we can use the coercivity of L on (Ker $L)^{\perp}$ (see (3.18)):

$$< Lr_i|r_i >\geq (1-\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}))\|r_i\|^2_{L^2},\ i=1,2.$$

By rescaling this inequality in the space variable, we obtain that:

$$< L^{\varepsilon}(w)|w >\geq \frac{1-\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})}{\varepsilon}\|w\|^2_{L^2}.$$

On the whole domain, we stick the previous estimates with a convenient system of cut-off functions by using the IMS formula (see [24]). We introduce the cut-off functions $\chi_0,\dots,\chi_N$ such that

- $\chi_i \in \mathcal{C}^{\infty}$
- supp $\chi_0 \in [-L,L]\setminus \bigcup_{i=1}^{N}[\sigma_i-\delta/2,\sigma_i+\delta/2]$
- supp $\chi_i \subset [\sigma_i-\frac{3\delta}{4},\sigma_i+\frac{3\delta}{4}]$ for $i\neq 0$ (4.17)
- $\sum_{i=0}^{N}(\chi_i)^2=1$

We can assume that there exists a constant K_δ, only depending on δ but not on $\sigma\in\Sigma_\delta$ such that

$$\|\chi_0'\|_{L^\infty}+\ldots+\|\chi_N'\|_{L^\infty}+\|\chi_0''\|_{L^\infty}+\ldots+\|\chi_N''\|_{L^\infty}\leq K_\delta. \tag{4.18}$$

We have:

$$< L^{\varepsilon}(w)|w >=\sum_{i=0}^{N}< L^{\varepsilon}(w)|\chi_i^2 w >$$

It is clear that

$$< -\frac{1}{\varepsilon}w_1e_1+f_\varepsilon^\sigma w|\chi_i^2w >=< -\frac{1}{\varepsilon}(\chi_iw)_1e_1+f_\varepsilon^\sigma(\chi_iw)|\chi_iw > .$$

In addition, we have:

$$\sum_{i=0}^{N}< -\partial_{xx}w|\chi_i^2w >=\sum_{i=0}^{N}(< -\partial_{xx}(\chi_iw)|\chi_iw > +2< \partial_x\chi_i\partial_xw|\chi_iw > + < w\partial_{xx}\chi_i|\chi_iw >) .$$

We remark that

$$2\sum_{i=0}^{N}< \partial_x\chi_i\partial_xw|\chi_iw >=\sum_{i=0}^{N}< \partial_xw\partial_x(\chi_i^2)|w >=< \partial_xw\partial_x(\sum_{i=0}^{N}(\chi_i^2))|w >=0$$

since $\sum_{i=0}^{N}\chi_i^2=1$.
Hence we obtain that

$$< L^{\varepsilon}(w)|w >=\sum_{i=0}^{N}< L^{\varepsilon}(\chi_iw)|\chi_iw > + < (\sum_{i=0}^{N}\chi_i\partial_{xx}\chi_i)w|w > .$$

We can then use the previous estimates for each term $< L^{\varepsilon}(\chi_iw)|\chi_iw >$ since χ_0w has its supports in the domains and since χ_iw has its support in the i^{th} wall. In addition, by (4.18), the additional term can be controlled:

$$\left|< (\sum_{i=0}^{N}\chi_i\partial_{xx}\chi_i)w|w >\right|\leq C\|w\|^2_{L^2}.$$

So we obtain that there exists c such that for all $\varepsilon > 0$, for all $\theta \in \mathbb{R}^N$, for all $\sigma \in \Sigma_\delta$, for all $w \in \mathcal{W}^\varepsilon_{\theta,\sigma}$,

$$< L^\varepsilon(w)|w > \geq \frac{1-c\varepsilon}{\varepsilon}\|w\|^2_{L^2}. \tag{4.19}$$

Using the previous estimate, one can obtain the following norms equivalence on $\mathcal{W}^\varepsilon_{\theta,\sigma}$:

$$\begin{aligned} c_1\left(\varepsilon\|\partial_{xx}w\|_{L^2} + \frac{1}{\varepsilon}\|w\|_{L^2}\right) \leq \|\mathbf{m}_\varepsilon \times L^\varepsilon w\|_{L^2} \leq c_2\left(\varepsilon\|\partial_{xx}w\|_{L^2} + \frac{1}{\varepsilon}\|w\|_{L^2}\right) \\ c_1\|w\|_\varepsilon \leq (< L^\varepsilon(w)|w >)^{\frac{1}{2}} \leq c_2\|w\|_\varepsilon. \end{aligned} \tag{4.20}$$

In addition we have:

$$\|\mathbf{m}_\varepsilon \times L^\varepsilon(w)\|^2_{L^2} \geq \frac{1-c\varepsilon}{\varepsilon} < L^\varepsilon(w)|w > . \tag{4.21}$$

The constants c_1, c_2 and c do not depend on $\theta \in \mathbb{R}^N$ and $\sigma \in \Sigma_\delta$.

Third step: variational estimates.

From the equivalence of norms (4.20), we estimate w by multiplying (4.13) by $L^\varepsilon(w)$. We obtain:

$$< \partial_t w|L^\varepsilon(w) > + \|\mathbf{m}_\varepsilon \times L^\varepsilon(w)\|^2_{L^2} = < a_\varepsilon + l_\varepsilon w|L^\varepsilon(w) > + < P_\varepsilon w|L^\varepsilon(w) > + < \mathcal{G}^\varepsilon(w,\theta,\sigma)|L^\varepsilon(w) > .$$

The first right hand side term is a small perturbation that does not raise any difficulty.

The last hand side term is estimated by (4.15) and (4.20):

$$|< \mathcal{G}^\varepsilon(w,\theta,\sigma)|L^\varepsilon(w) >| \leq C\|w\|_\varepsilon\|\mathbf{m}_\varepsilon \times L^\varepsilon(w)\|^2_{L^2}. \tag{4.22}$$

The first left hand side term yields:

$$\begin{aligned} < \partial_t w|L^\varepsilon(w) > &= \frac{1}{2}\frac{d}{dt} < L^\varepsilon(w)|w > -\frac{1}{2} < \partial_t f^\sigma_\varepsilon w|w > \\ &= \frac{1}{2}\frac{d}{dt} < L^\varepsilon(w)|w > -\frac{1}{2}\sum_{i=1}^N \partial_t\sigma_i < \partial_{\sigma_i} f^\sigma_\varepsilon w|w > \end{aligned}$$

Therefore we obtain that

$$\frac{1}{2}\frac{d}{dt} < L^\varepsilon(w)|w > + \|\mathbf{m}_\varepsilon \times L^\varepsilon(w)\|^2_{L^2} \leq \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}) + (\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}) + C\|w\|_\varepsilon)\|\mathbf{m}_\varepsilon \times L^\varepsilon(w)\|^2_{L^2} + |A(w)|, \tag{4.23}$$

where

$$A(w) = < P_\varepsilon w|L^\varepsilon(w) > + \frac{1}{2}\sum_{i=1}^N \partial_t\sigma_i < \partial_{\sigma_i} f^\sigma_\varepsilon w|w > . \tag{4.24}$$

Estimate for $A(w)$.
As for the coercivity of L^ε, we estimate $A(w)$ for w with support in the domains, for w with support in one wall, and we generalize the obtained estimates for a general w by using the IMS formula.

In the domains, $\partial_{\sigma_i} f^\sigma_\varepsilon = 0$, $L^\varepsilon(w)$ reduces to $-\varepsilon\partial_{xx}w + \frac{1}{\varepsilon}w$. In addition,

$$< P_\varepsilon w|L^\varepsilon(w) > = < \frac{h}{\varepsilon}w| - \varepsilon\partial_{xx}w + \frac{1}{\varepsilon}w >$$

so

$$A(w) \leq \frac{\|h\|_{L^\infty}}{\varepsilon}\|w\|_{L^2}\|\mathbf{m}_\varepsilon \times L^\varepsilon(w)\|_{L^2} \leq \|h\|_{L^\infty}\|\mathbf{m}_\varepsilon \times L^\varepsilon(w)\|^2_{L^2}.$$

Concerning the walls we assume that supp $w \subset [\sigma_i - \delta, \sigma_i + \delta]$. As for the coercivity of L^ε, we describe w in the mobile frame attached to $\mathbf{m}_\varepsilon$ using (4.16), and we obtain that, in the unknown (r_1, r_2), the main part of $A(w)$ writes

$$< \frac{h}{\varepsilon} \ell r | L r >$$

so by rescaling (3.20), we obtain that for w of support in the i^{th} wall:

$$|A(w)| \leq \|h\|_{L^\infty} \|\mathbf{m}_\varepsilon \times L^\varepsilon(w)\|_{L^2}^2.$$

By using the relevant cut off functions χ_i satisfying (4.17) and the IMS formula, we obtain that

$$|A(w)| \leq (h_0 + C\sqrt{\varepsilon}) \|\mathbf{m}_\varepsilon \times L^\varepsilon(w)\|_{L^2}^2. \tag{4.25}$$

End of the proof.
From (4.23), since $< L^\varepsilon(w)|w >$ controls $\|w\|_\varepsilon$ (see estimates (4.19)), with the previous estimates, we obtain that while σ remains in Σ_δ:

$$\frac{1}{2}\frac{d}{dt} < L^\varepsilon(w)|w > + \|\mathbf{m}_\varepsilon \times L^\varepsilon(w)\|_{L^2}^2 \left(1 - h_0 - c\sqrt{\varepsilon} - C\,(< L^\varepsilon(w)|w >)^{\frac{1}{2}}\right) \leq \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}).$$

For ε small enough, $1 - h_0 - c\sqrt{\varepsilon} \geq \dfrac{1-h_0}{2}$. Then while $< L^\varepsilon(w)|w > (t) \leq \dfrac{1-h_0}{4C}$, we have:

$$\frac{1}{2}\frac{d}{dt} < L^\varepsilon(w)|w > + \frac{1-h_0}{4} \|\mathbf{m}_\varepsilon \times L^\varepsilon(w)\|_{L^2}^2 \leq \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}),$$

and by (4.21) there exists $\gamma > 0$ such that while $< L^\varepsilon(w)|w > (t) \leq \frac{1-h_0}{4C}$,

$$\frac{d}{dt} < L^\varepsilon(w)|w > + \frac{\gamma}{\varepsilon} < L^\varepsilon(w)|w > \leq \mathcal{O}(e^{-\frac{\delta}{4\varepsilon}}).$$

By comparison arguments, if ε is small enough so that $\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})$ is small, we obtain that while $< L^\varepsilon(w)|w > (t) \leq \dfrac{1-h_0}{4C}$,

$$< L^\varepsilon(w)|w > (t) \leq \frac{\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})}{\gamma} + e^{-\frac{\gamma t}{\varepsilon}} \left(< L^\varepsilon(w_0)|w_0 > - \frac{\mathcal{O}(e^{-\frac{\delta}{4\varepsilon}})}{\gamma} \right) \tag{4.26}$$

so that if $< L^\varepsilon(w_0)|w_0 >$ is small enough, then $< L^\varepsilon(w)|w > (t)$ remains less than $\dfrac{1-h_0}{4C}$, and Estimate (4.26) remains valid for all time. So the only condition for the validity of these estimates is that $\sigma(t)$ remains in Σ_δ.

The previous estimate shows that the trajectory of the Landau-Lifschitz system (4.1) with initial data m_0 in a small neighborhood of $\mathcal{M}_{2\delta}$ remains in a small neighborhood of $\mathcal{M}_\delta$ while it does not reach the boundary of $\mathcal{M}_\delta$, *i.e.* while the walls are not collapsing.

We control this non collapsing on time intervals in which $|\sigma(t) - \sigma^{ref}(t)| \leq \delta$ (since by assumption, $\sigma^{ref}(t)$ remains in $\Sigma_{2\delta}$). Using that the system (3.5) satisfied by (θ, σ) is a small perturbation of the system (4.7) satisfied by $(\theta^{ref}, \sigma^{ref})$, since the size of this perturbation is controlled by the size of w estimated by (4.26), we obtain that $(\theta(t), \sigma(t))$ remains close to $(\theta^{ref}(t), \sigma^{ref}(t))$ on time intervals of size $\mathcal{O}(e^{\frac{c}{\varepsilon}})$, and we conclude the proof of Theorem 4.1.

5 Conclusion, open problems

5.1 Straight round nanowire model

The dynamics for Equation (4.1) is the following. Starting from any initial data in $H^1([0, L]; S^2)$, we observe a first very short phase in which the magnetization organizes itself in domains and walls.

In a second exponentially long phase, the motion of the walls is approximatively governed by the system (4.7). This phase is well described by our Theorem 4.1. On the other hand, the first phase is not mathematically understood. In addition, we are not able to describe the collapse of two walls, or the collapse of a wall with the boundary. This phenomenon can be induced by the applied magnetic field (by relaxing the assumptions on the applied field) or can occur "naturally" without applied field when two walls are to close one to one another. This kind of dynamics is described by Chen for the Allen Cahn model (see [23]).

5.2 Other geometries of nanowires

For non round nanowires, the common model is to add an anisotropy in the equivalent demagnetizing field, that is setting:

$$H_d(m) = -\alpha m_2 e_2 - \beta m_3 e_3, \quad \alpha > 0, \beta > 0.$$

This model can be justified as in Section 2 by considering the limit when η tends to zero of the Landau-Lifschitz equation on the domain $[0, L] \times \eta\omega$ where $\partial\omega$ is an ellipse:

$$\omega = \left\{ (y, z), \ \frac{y^2}{a^2} + \frac{z^2}{b^2} < 1 \right\}.$$

Let us assume that $0 < \alpha < \beta$. The corresponding demagnetizing energy writes:

$$\mathcal{E}_{dem}(m) = 2 \int_{\mathbb{R}} \left(\alpha |m_2|^2 + \beta |m_3|^2 \right)$$

so that the energy of a wall is minimum when the wall profile takes its values in the plane $0xy$, thus we lose the invariance by rotation around the wire axis. This induces a very different behavior compared to the walls motion in a round wire. Indeed, walls dynamics presents two different regimes according to the value of the applied field. There exists a threshold h_s such that for small constant applied field h with $|h| < h_s$, the motion of the wall is described by an exact solution of the form

$$R_\theta \left(M^0 (\frac{x - ct}{\delta}) \right)$$

where θ, c and δ only depend on h, so that the wall profile does not turn around the wire and is dilated (compare with the exact profile (3.6) given for a round wire). The stability for this kind of motion can be proved with the same method as for the round wire. The problem is much more complicated for great applied field. If $|h| \geq h_s$, then we observe a dilatation translation and rotation of the wall of the form:

$$R_{\theta(t)} \left(M^0 (\frac{x - X(t)}{\delta(t)}) \right)$$

where the velocity $\dot{X}(t)$, the dilatation rate $\delta(t)$ and the rotation speed $\dot{\theta}(t)$ are periodic in time (while they are constant for round wires). This behavior is observed numerically and in the experimentations. It is described in the literature in physics (with the same kind of calculation as in [50]), but the key point of the argumentation is that a term is small so it is neglected in the calculations. This approximation is not mathematically justified and the existence of exact solutions describing such a behavior is an open problem.

For the applications, it would be interesting to study the effects of the curvature on the walls motion for non straight nanowires. In the experimentations, we can see that walls prefer strong curvatures. Even in short finite curved wires, a wall located at the maximum of the wire curvature seems to be stable while a single wall on a short straight wire is unstable (see Section 3). The understanding of this behavior is totally open.

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