On the finite element solution of helmholtz problems in anisotropic media

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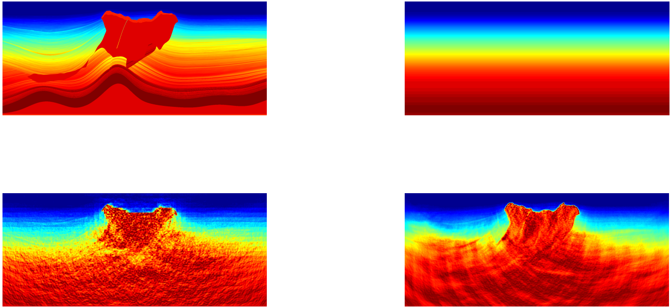
FWI using DG or FD

Same acoustic model of size 35×15 km. Exact same FWI algorithm (n iterations, frequencies, ...), no initial information.



Computations performed by Florian Faucher (Magique-3D)

The time harmonic anisotropic scalar wave equation

$$\begin{cases} \nabla \cdot \Big(A(\mathbf{x})\nabla u(\mathbf{x})\Big) + \mu(\mathbf{x})u(\mathbf{x}) = 0 & \text{in } \Omega \\ u(\mathbf{x}) = g(\mathbf{x}) & \text{on } \partial\Omega, \end{cases}$$

Anisotropies are handled inside the divergence operator

The matrix A is symmetric

$A = \lambda \mathbf{I}$, for isotropic media

A and μ are often piecewise constant.

The domain Ω is 2D or 3D with boundary $\partial\Omega$.

Other boundary conditions can be as well considered

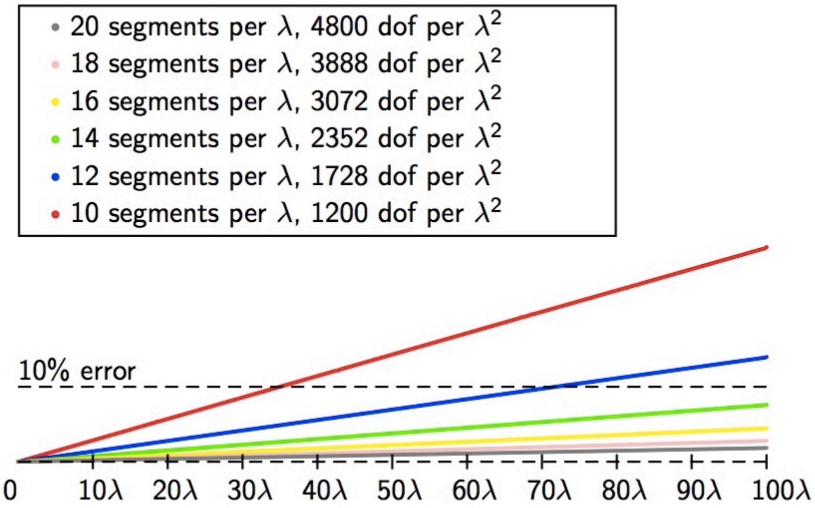
Motivation: Computations on very large domains

Ω is very large vs the wavelength

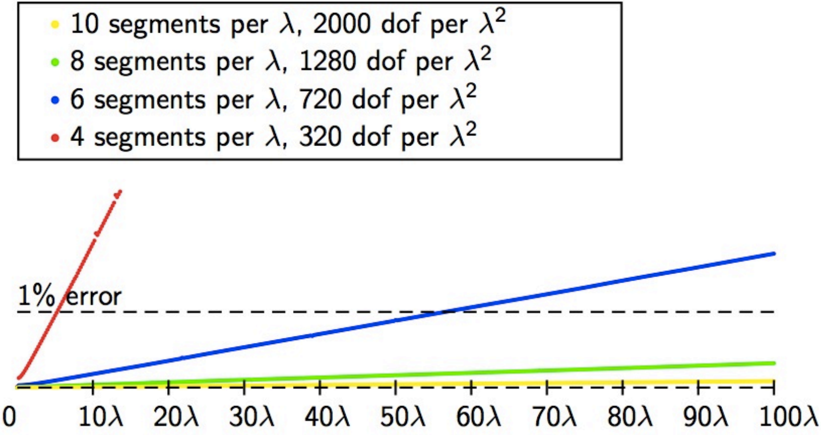
Need to augment the density of nodes to maintain a given level of accuracy

*Babuska, SA Sauter, Is the pollution effect of the FEM avoidable for the Helmholtz equation considering high wave numbers? SIAM Journal on numerical analysis, 1997 (**cited 476 times**)*

Error at every wavelength in $\mathbb{P}_2$ for 10, 12, 14, 16, 18 segments per λ



Error at every wavelength in $\mathbb{P}_3$ for 4, 6, 8, 10 segments per λ



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Exceed the storage capacities.

Discontinuous Galerkin methods do resist better to pollution effect

General setting

Studies show that DG weak inter-element continuity contributes to fight the pollution effect

But DG approximations imply to increase the number of nodes significantly

Lead to Trefftz methods

In particular, Ultra-Weak-Variational-Formulations proposed by B. Després and O. Cessenat.

Trefftz method: shape functions are solutions to the problem

Set on a single element T: Trefftz formulation reduces to the boundary of the element

Rewrite the IPDG and UWVF formulation in a different context

Classically, plane wave bases or Bessel functions inside each element.

Here we use an auxiliary method to compute local solutions.

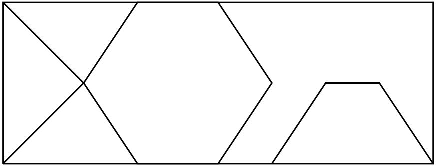
We call these methods

BEM-STDG	**BEM-UWVF**
FEM-STDG	**FEM-UWVF**
FD-STDG	**FD-UWVF**
Plane-wave-STDG	**Plane-wave-UWVF**

General setting

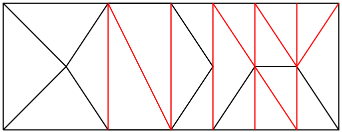
- For isotropic media
  - Boundary Integral Equations (BIE) lead to less pollution effect than FEMs
  - Recently, Hofreither et al. (2015) have proposed a FEM in which local shape functions are obtained on the basis of a BIE.
  - In the same spirit, we propose a DG method using local shape solutions to the Helmholtz problem that are matched at the interface of the mesh thanks to the Dirichlet-to-Neumann (DtN) operator which is computed with a BIE.
- For anisotropic media
  - Very precise Finite Element Method to reproduce the BIE
  - Can also handle strongly heterogeneous media

The mesh

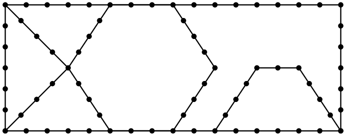


A smart finite element method

The mesh



A classical finite element mesh



A smart finite element mesh

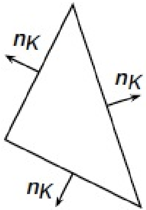
Trial and test functions

Trial and test functions are solutions to the Helmholtz equation in each element

$$\operatorname{div}\Big(A_K \nabla u_K(\mathbf{x})\Big) + \mu_K u_K(\mathbf{x}) = 0 \text{ in } K$$

u_K is uniquely defined by its Dirichlet trace if K is small enough (geometrical criterion)

$$u_K \in H^{1/2}(\partial K)$$



The discrete variational space is then obtained by considering a discrete trace space

$$u_K \text{ is } \mathbb{P}_r\text{-continuous on } \partial K.$$

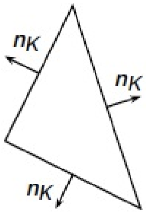
Trial and test functions

Trial and test functions are solutions to the Helmholtz equation in each element

$$\mathrm{div}\Big(A_K \nabla u_K(\mathbf{x})\Big) \; + \; \mu_K u_K(\mathbf{x}) \; = \; 0 \text{ in } K$$

u_K is uniquely defined by its Robin trace

$$A_k \nabla u_K \cdot n_K + i\eta u_K \in H^{-1/2}(\partial K)$$



The discrete variational space is then obtained by considering a discrete trace space

$$A_k \nabla u_K \cdot n_K + i\eta u_K \text{ is } \mathbb{P}_r\text{-discontinuous on } \partial K.$$

The reciprocity principle for anisotropic media

Since u and v are solutions of the anisotropic Helmholtz equation on each element K.

$$\int_{\partial K} (A\nabla u) \cdot nv \; = \; \int_K (A\nabla u) \cdot \nabla v - \kappa^2 uv = \int_{\partial K} u \cdot n(A\nabla u) \tag{1}$$

The reciprocity principle is the main ingredient of any Trefftz method

$$\int_{\partial K} uq - pv \; = \; 0$$

Denoting by

$$p = (A\nabla u) \cdot n \quad q = (A\nabla v) \cdot n$$

This expression is the main ingredient of all the Trefftz formulation.

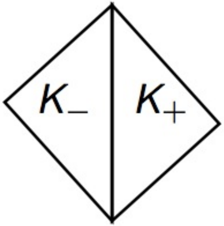
Symmetric Trefftz Variational Formulation

Summing over all the elements

$$0 = \sum_K \int_{\partial K} uq - pv ds_{\mathbf{x}}$$

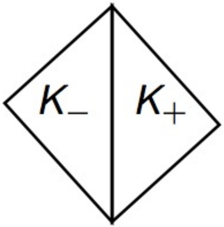
with the collection of interior edges Γ and $\partial\Omega$ the boundary of the domain

$$\begin{cases} 0 & = \displaystyle\int_\Gamma u_+q_+ + u_-q_- - p_+v_+ + p_-v_- ds_{\mathbf{x}} \\ & + \displaystyle\int_{\partial\Omega} uq - pv ds_{\mathbf{x}} \end{cases}$$



Symmetric Trefftz Variational Formulation

$$\begin{cases} 0 & = \displaystyle\int_\Gamma u_+ q_+ + u_- q_- - p_+ v_+ + p_- v_- ds_{\mathbf{x}} \\ & + \displaystyle\int_{\partial\Omega} uq - pv ds_{\mathbf{x}} \end{cases}$$



The exact solution satisfies $u_+ = u_-$ and $p_+ = -p_-$ on Γ

$$\begin{cases} 0 = \displaystyle\int_\Gamma u_- q_+ + u_+ q_- + p_- v_+ + p_+ v_- ds_{\mathbf{x}} \\ \quad + \displaystyle\int_{\partial\Omega} uq - pv ds_{\mathbf{x}} \end{cases}$$

and $u = g$ on $\partial\Omega$

$$\underbrace{\int_\Gamma u_- q_+ + u_+ q_- + p_- v_+ + p_+ v_- ds_{\mathbf{x}} - \int_{\partial\Omega} pv + uq ds_{\mathbf{x}}}_{a(u,p;v,q)} = \underbrace{-2\int_{\partial\Omega} gq ds_{\mathbf{x}}}_{\ell_2(v)}$$

The symmetric variational formulation

Adding the penalization terms ($[u] = 0$ on Γ and $u = g$ on $\partial\Omega$):

$$\underbrace{\int_\Gamma \alpha[u][v] + \int_{\partial\Omega} \alpha uv}_{b(u,p;v,q)} = \underbrace{\int_{\partial\Omega} \alpha g_D v}_{\ell_2(v)}$$

This leads to the Trefftz-DG formulation

$$a(u, p; v, q) + b(u, p; v, q) = \ell_1(q) + \ell_2(v).$$

Why the symmetry is important ?

for the linear algebra solver: it needs less memory

it has been observed that BIE methods are more stable.

Now, the unknowns are u and $p = A\nabla u \cdot \mathbf{n}$ on each face of the mesh. One may be removed.

The DtN operator

Let

$$u_K(\mathbf{x}) \text{ be given on } \partial K.$$

The Neumann trace

$$p_K \;=\; A_K\, \nabla u_K \cdot \mathbf{n}_K \text{ on } \partial K$$

may then be deduced thanks to the Dirichlet-to-Neumann operator

$$DtN : \begin{cases} H^{\frac{1}{2}}(\partial K) \longrightarrow H^{-\frac{1}{2}}(\partial K) \\ u_K \longmapsto p_K \end{cases}$$

and we end up with a system involving unknowns defined on the boundary of each element.

The problem to be addressed: compute the DtN operator

The DtN operator approximation: include an auxiliary numerical method

We can think about different methods for isotropic media like:

- finite element/finite difference method based on the velocity/pressure formulation
- Boundary element method

Why BEM? They do resist very well to pollution effect

The secondary numerical method: BEM

$$\frac{V_K p_K}{\lambda_K} = \frac{M_K u_K}{2} - N_K u_K$$

u_K is approximated by a $\mathbb{P}_r$-continuous function

p_K is approximated by a $\mathbb{P}_{r'}$-discontinuous function

V_K and N_K are the single layer and double layer operators.

$$\begin{cases} (M_K u_K, q_k)_{\partial K} & = \displaystyle\int_{\partial K} u_K(\mathbf{x})\ q_K(\mathbf{x}) ds_{\mathbf{x}}, \\ (V_K p_K, q_k)_{\partial K} & = \displaystyle\int_{\partial K}\int_{\partial K} p_K(\mathbf{x})\ G_K(\mathbf{x}-\mathbf{y})\ q_K(\mathbf{y}) ds_{\mathbf{x}} ds_{\mathbf{y}}, \\ (N_K u_K, q_k)_{\partial K} & = \displaystyle\int_{\partial K}\int_{\partial K} p_K(\mathbf{x})\ \frac{\partial G_K}{\partial \mathbf{n}_{\mathbf{y}}}(\mathbf{x}-\mathbf{y}) q_K(\mathbf{y}) ds_{\mathbf{x}} ds_{\mathbf{y}} \end{cases}$$

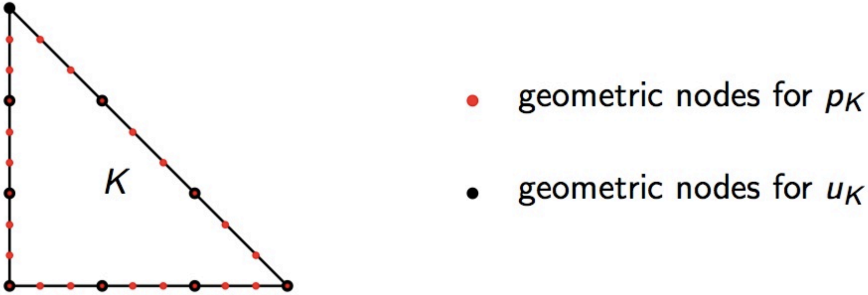
with

$$G(\mathbf{x}) = \frac{\exp(ik_K\|\mathbf{x}\|)}{4\pi\|\mathbf{x}\|} \text{ with } k_K = \sqrt{\frac{\mu_K}{\lambda_K}}$$

The boundary element method

u_K is approximated by a $\mathbb{P}_r$-continuous function

p_K is approximated by a $\mathbb{P}_{r'}$-discontinuous function



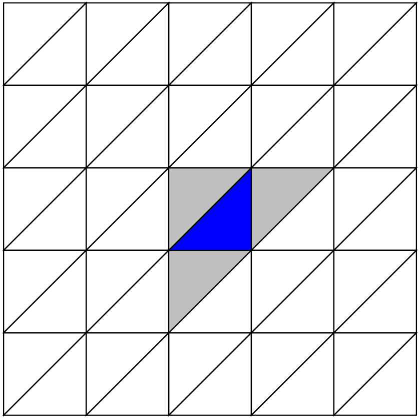
We can use different meshes for u_K and p_K.

Idea: the Neumann trace must be computed accurately

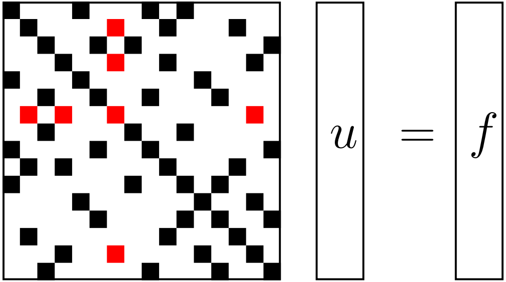
Remark: p_K is discontinuous, only at the geometric singularities.

The skeleton of the matrix

Connection of the elements



The final formulation

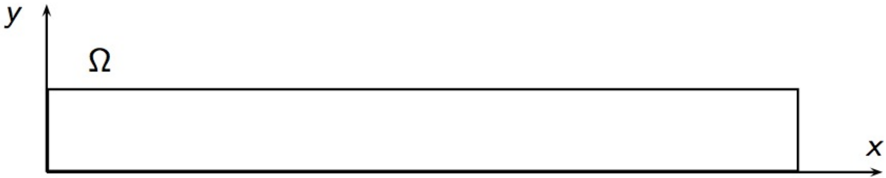


Symmetric block sparse matrix

full small blocks

really adapted to GMRES Solver

A numerical simulation



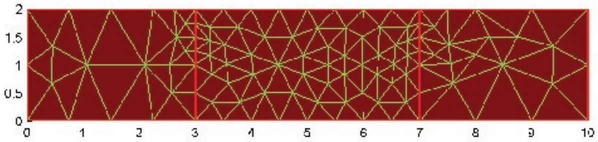
$$\begin{cases} \Delta u(\mathbf{x}) \ + \ k^2 \ u(\mathbf{x}) \ = \ 0 & \text{in } \Omega \\ u(\mathbf{x}) = 1 & \text{at } x = 0, \\ \dfrac{\partial u}{\partial n}(\mathbf{x}) = 0 & \text{at } \mathbf{x} = N\lambda \\ \frac{\partial u}{\partial n}(\mathbf{x}) + iku(\mathbf{x}) = 0 & \text{at } y = 0 \text{ and } N\lambda \end{cases} \tag{2}$$

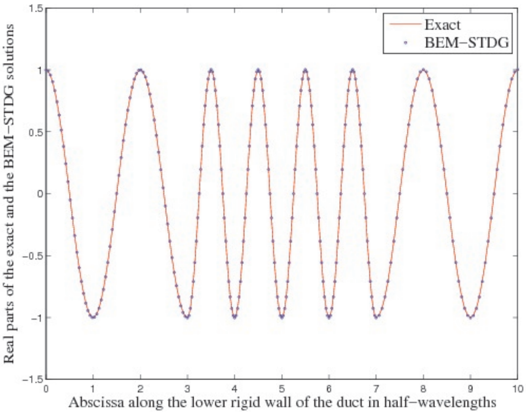
Poly degree	nodes per λ	Method	Error at 175λ for $\mathbb{P}_2$ Error at 500λ for $\mathbb{P}_3$
$m = 2$	12	IPDG	72 %
		BEM-STDG	22 %
	16	IPDG	67 %
		BEM-STDG	5.6 %
	24	IPDG	13 %
		BEM-STDG	0.8 %
$m = 3$	12	IPDG	19 %
		BEM-STDG	1.6 %
	18	IPDG	1.7 %
		BEM-STDG	0.1 %
	24	IPDG	0.3 %
		BEM-STDG	0.02 %

Degree	Density (nodes/λ)	Method	Condition number 50 λ	Condition number 500 λ	CPU time 500 λ
$m=2$	12	IPDG	$8.8\,10^{10}$	$4.7\,10^{11}$	2.54
		BEM-STDG	$6.06\,10^{7}$	$1.00\,10^{8}$	4.76
	24	IPDG	$1.2\,10^{12}$	$2.76\,10^{12}$	19.03
		BEM-STDG	$5.96\,10^{8}$	$9.42\,10^{9}$	8.5
$m=3$	12	IPDG	$4.2\,10^{11}$	$8.9\,10^{11}$	2.13
		BEM-STDG	$2.1\,10^{8}$	$1.5\,10^{9}$	4.75
	24	IPDG	$2.0\,10^{12}$	$6.2\,10^{13}$	20.81
		BEM-STDG	$9.52\,10^{8}$	$8.07\,10^{10}$	8.4
$m=4$	8	IPDG	$1.43\,10^{11}$	$3.78\,10^{11}$	0.66
		BEM-STDG	$1.13\,10^{8}$	$1.08\,10^{8}$	3.89
	24	IPDG	$2.38\,10^{12}$	$2.41\,10^{14}$	17.91
		BEM-STDG	$1.7\,10^{9}$	$1.7\,10^{11}$	8.41

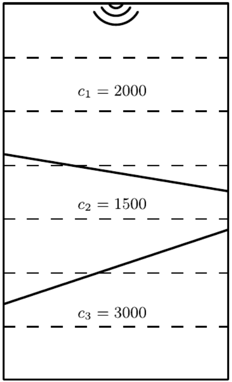
Condition number and CPU time for h p refinements

Case of an unstructured mesh



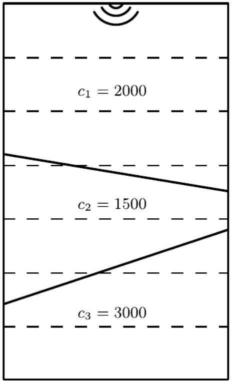


Case of an unstructured mesh of an heterogenous medium

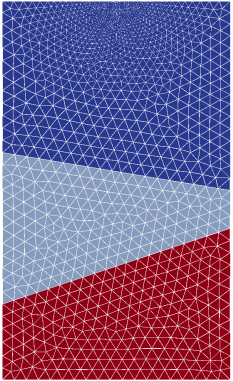


Configuration

Case of an unstructured mesh of an heterogenous medium

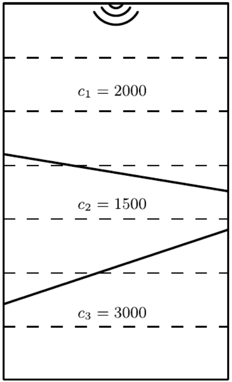


Configuration



Mesh

Case of an unstructured mesh of an heterogenous medium



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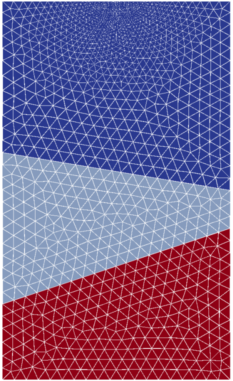
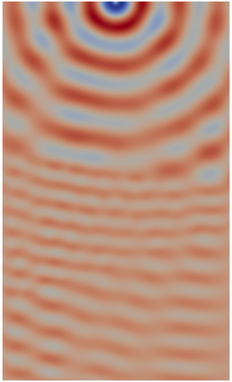
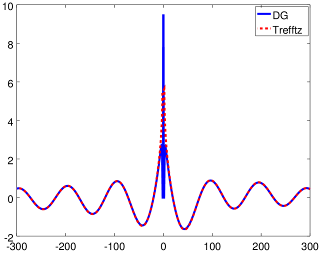


Mesh

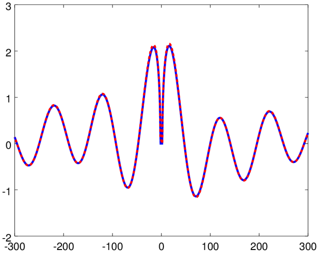


DG Solution
(Imaginary part)

Case of an unstructured mesh of an heterogenous medium



Real part



Imaginary part

The anisotropic Ultra Weak Variational method

The basis of the reciprocity principle is the reciprocity principle

$$\sum_K \int_{\partial K} uq - pv \; = \; 0$$

and the identity

$$uq - pv \; = \; \frac{\big(p + i\eta u\big)\big(q - i\eta v\big) - \big(p - i\eta u\big)\big(q + i\eta v\big)}{2i\eta}$$

The choice of η

- is well known for isotropic and homogenous media $\eta = k$
- has not been defined correctly for hetrogeneous media
- has not yet been defined for heterogeneous anisotropic media

The trial and test functions

We denote by

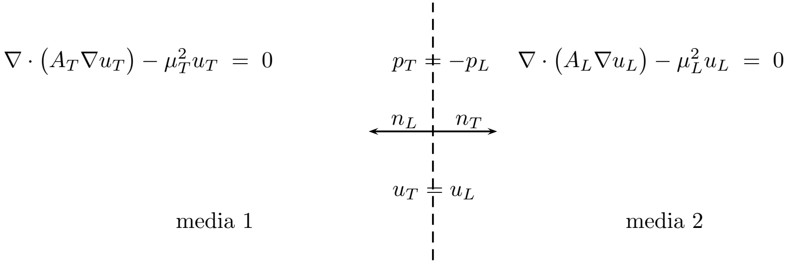
$$\begin{cases} x_T = & = \frac{u_T}{2} + \frac{p_T}{2i\eta_T}, \\ \widetilde{x}_T & = \frac{u_T}{2} - \frac{p_T}{2i\eta_T}, \\ y_T & = \frac{v_T}{2} + \frac{q_T}{2i\eta_T}, \\ \widetilde{y}_T & = \frac{v_T}{2} - \frac{q_T}{2i\eta_T}. \end{cases}$$

The reciprocity principle:

$$\int_{\partial T} x_T \widetilde{y_T} \eta_T ds = \int_{\partial K} y_T \widetilde{x}_T \eta_T ds$$

x_T is a trial function

Reflexion transmission between two media



We consider a plane wave orthogonal to the boundary

$$\begin{cases} u_T(x) = I_T^{in} \exp(ik_T x_1) + I_T^{out} \exp(-ik_T x_1) & x_1 > 0, \\ u_L(x) = I_L^{in} \exp(-ik_L x_1) + I_L^{out} \exp(ik_L x_1) & x_1 < 0 \end{cases}$$

Link with the choice of η for the UWVF method

$$\begin{cases} I_T^{in} & = & \dfrac{u_T}{2} + \dfrac{p_T}{2i\eta_T} \\ I_T^{out} & = & \dfrac{u_T}{2} - \dfrac{p_T}{2i\eta_T} \\ I_L^{in} & = & \dfrac{u_L}{2} + \dfrac{p_L}{2i\eta_L} \\ I_L^{out} & = & \dfrac{u_L}{2} - \dfrac{p_L}{2i\eta_L} \end{cases}$$

with η given by

$$\begin{cases} \eta_T & = & k_T\sqrt{\mathbf{n}\cdot(A^T\mathbf{n})} \\ \eta_L & = & k_L\sqrt{\mathbf{n}\cdot(A^L\mathbf{n})} \end{cases}$$

This will be our choice of η

Transmission condition between two media

For the plane wave solution

$$I_T^{out} = R_{TL} I_T^{in} + T_{LT} I_L^{in},$$

$$I_L^{out} = R_{LT} I_L^{in} + T_{TL} I_T^{in}.$$

with

$$\begin{cases} R_{TL} = \frac{\eta_T - \eta_L}{\eta_L + \eta_T} & R_{LT} = \frac{\eta_L - \eta_T}{\eta_T + \eta_L} \\ T_{TL} = \frac{2\eta_T}{\eta_L + \eta_T} & T_{LT} = \frac{2\eta_L}{\eta_L + \eta_T} \end{cases}$$

The Transmission Condition between two media

This last expression is rephrased in terms of the x_T variables

$$\begin{cases} \widetilde{x}_T = R_T x_T + T_{LT} x_L, \\ \widetilde{x}_L = R_L x_L + T_{TL} x_T. \end{cases}$$

and in terms of u and p

$$\begin{cases} -p_T + i\eta_L u_T = p_L + i\eta_L u_L, \\ -p_L + i\eta_T u_L = p_T + i\eta_T u_T. \end{cases}$$

This is rather similar to an upwind flux.

The boundary conditions

$$\begin{cases} u & = & g_D, & \text{(Dirichlet condition)} \\ p & = & g_N & \text{(Neumann condition)} \end{cases}$$

can be rewritten in terms of incoming and outgoing waves

$$-\frac{1}{2i\eta}(A\boldsymbol{\nabla} u \cdot \boldsymbol{n} - i\eta u) = \frac{Q}{2i\eta}(A\boldsymbol{\nabla} u \cdot \boldsymbol{n} + i\eta u) + g$$

with Q given here by

$$\begin{cases} Q = -1, \quad g = g_D, & \text{on } \partial\Omega_D, \\ Q = +1, \quad g = -g_N/i\eta, & \text{on } \partial\Omega_N, \end{cases}$$

The boundary conditions

$$\begin{cases} u &= & g_D, & \text{(Dirichlet condition)} \\ p &= & g_N & \text{(Neumann condition)} \end{cases}$$

can be rewritten in terms of incoming and outgoing waves

$$\widetilde{x}_T = Qx_T + g$$

with Q given here by

$$\begin{cases} Q = -1, \quad g = g_D, & \text{on } \partial\Omega_D, \\ Q = +1, \quad g = -g_N/i\eta, & \text{on } \partial\Omega_N, \end{cases}$$

UWVF formulation with outgoing and incoming waves

The starting point is the reciprocity principle

$$\int_{\partial T} x_T \ \widetilde{y}_T \ \eta_T ds = \int_{\partial T} \widetilde{x}_T y_T \ \eta_T ds$$

UWVF formulation with outgoing and incoming waves

The starting point is the reciprocity principle

$$\int_{\partial T} x_T \ \widetilde{y}_T \ \eta_T ds = \sum_{F_T \in \mathcal{F}_\partial} \int_{F_T} \widetilde{x}_T y_T \ \eta_T ds + \sum_{F_{TL} \in \mathcal{F}_\mathcal{I}} \int_{F_{TL}} y_T \widetilde{x}_T \ \eta_T ds$$

where we have decomposed the boundary into two parts

- the edges shared $\mathcal{F}_\mathcal{I}$ with another element L
- the exterior edges $\mathcal{F}_\partial$

UWVF formulation with outgoing and incoming waves

The starting point is the reciprocity principle

$$\int_{\partial T} x_T \; \widetilde{y}_T \;\; \eta_T ds = \sum_{F_T \in \mathcal{F}_\partial} \int_{F_T} \widetilde{x}_T y_T \;\; \eta_T ds +$$

$$\sum_{F_{TL} \in \mathcal{F}_\mathcal{I}} \int_{F_{TL}} T_{TL} x_L y_T \;\; \eta_T ds + \sum_{F_{TL} \in \mathcal{F}_\mathcal{I}} \int_{F_{TL}} R_{TL} x_T y_T \;\; \eta_T ds$$

For the edges of the boundary, we replace $\widetilde{x}_T$ by

$$\widetilde{x}_T = Q x_T + g$$

UWVF formulation with outgoing and incoming waves

The starting point is the reciprocity principle

$$\int_{\partial T} x_T \ \widetilde{y}_T \ \eta_T ds = \sum_{F_T \in \mathcal{F}_\partial} \int_{F_T} (Q_T x_T + g) y_T \ \eta_T ds +$$
$$\sum_{F_{TL} \in \mathcal{F}_\mathcal{I}} \int_{F_{TL}} T_{TL} x_L y_T \ \eta_T ds + \sum_{F_{TL} \in \mathcal{F}_\mathcal{I}} \int_{F_{TL}} R_{TL} x_T y_T \ \eta_T ds$$

For the edges of the boundary, we replace $\widetilde{x}_T$ by

$$\widetilde{x}_T = Q x_T + g$$

For the interior edges, we replace $\widetilde{x}_T$ by

$$\widetilde{x}_T = R_{TL} x_T + T_{LT} x_L.$$

It remains to understant what is the $\widetilde{\cdot}$ operator

The incoming to outgoing operator

Given $x_T \in L^2(\partial K)$, we consider the solution of

$$\left\{ \begin{array}{llll} \text{Find } u \in H^1(K) \text{ tel que} \nabla \cdot \big(A\nabla u\big) + \mu u & = & 0 & \text{dans } T \\ \dfrac{u_T}{2} - \dfrac{A\nabla u_T \cdot \mathbf{n}_T}{2i\eta_T} = x_T & & & \text{sur } \partial T \end{array} \right.$$

$$\widetilde{x}_T = \frac{u_T}{2} + \frac{p_T}{2i\eta_T} = u_T - \left(\frac{u_T}{2} - \frac{p_T}{2i\eta_T}\right) = u_T - x_T.$$

No need to compute p_T to get $\widetilde{x}_T$.

The incoming to outgoing operator is a unitary operator

$$\left\{ \begin{array}{llll} \widetilde{\cdot}_T : & L^2(\partial T) & \longrightarrow & L^2(\partial T) \\ & x_T & \longmapsto & \widetilde{x}_T \end{array} \right.$$

$$\begin{cases} \nabla \times \dfrac{1}{\mu} \nabla \times \mathbf{E} - \varepsilon k^2 \mathbf{E} = 0 & \quad \textit{in} \quad \Omega \\ \mathbf{n} \times (\dfrac{1}{\mu} \nabla \times \mathbf{E}) - Z \mathbf{n} \times (\mathbf{E} \times \mathbf{n}) = \mathbf{g_R} \quad \textit{on} \quad \partial\Omega_R \\ \mathbf{n} \times (\mathbf{E} \times \mathbf{n}) = \mathbf{g_D} \quad \textit{on} \quad \partial\Omega_D \end{cases}$$

Problem well posed.

$\mathbf{E} \in H(curl, \Omega)$

ε, $\mu > 0$, piece-wise constant

$\mathbf{g_D}, \mathbf{g_R} \in L^2_t(\partial\Omega)$

$\mathrm{Im}(Z) > 0$

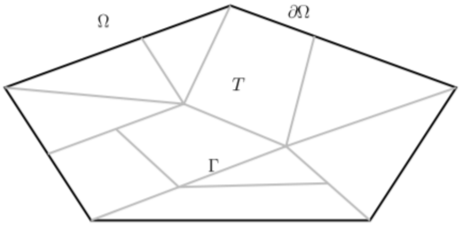
$k = \frac{\omega}{c}$

The symmetric DG Trefftz formulation

The DG formulation mesh

$$\overline{\Omega} = \bigcup_{T \in \mathcal{T}} \overline{T}, \quad T \cap K = \emptyset \quad \text{if} \quad T \neq K$$

$$\Gamma = \bigcup_{F \in \mathcal{F}_I} F, \qquad \partial\Omega = \bigcup_{F \in \mathcal{F}_\partial} F$$



The symmetric DG Trefftz formulation

The basis function space

$$X_T = \Big\{ \boldsymbol{\omega} \in H(curl, T); \quad \nabla \times \frac{1}{\mu_T} \nabla \times \boldsymbol{\omega} - \varepsilon_T k_T^2 \boldsymbol{\omega} = 0 \quad \text{in} \quad T,$$

$$\mathbf{n} \times (\boldsymbol{\omega}|_{\partial T} \times \mathbf{n}), \quad \mathbf{n} \times (\frac{1}{\mu} \nabla \boldsymbol{\omega})|_{\partial T} \in L^2(\partial T) \Big\}$$

The symmetric DG Trefftz formulation

A Green formula on one element T:

$$\int_T (\nabla \times \frac{1}{\mu_T} \nabla \times \mathbf{E}) \cdot \boldsymbol{\omega} dx = \int_T (\frac{1}{\mu_T} \nabla \times \mathbf{E}) \cdot (\nabla \times \boldsymbol{\omega}) dx + \int_{\partial T} \mathbf{n} \times (\frac{1}{\mu_T} \nabla \times \mathbf{E})|_{\partial\Omega} \cdot \boldsymbol{\omega} ds$$

The symmetric DG Trefftz formulation

A Green formula on one element T:

$$\int_T (\nabla \times \frac{1}{\mu_T} \nabla \times \mathbf{E}) \cdot \boldsymbol{\omega} dx = \int_T (\frac{1}{\mu_T} \nabla \times \mathbf{E}) \cdot (\nabla \times \boldsymbol{\omega}) dx + \int_{\partial T} \mathbf{n} \times (\frac{1}{\mu_T} \nabla \times \mathbf{E})|_{\partial \Omega} \cdot \boldsymbol{\omega} ds$$

Replace $\nabla \times \frac{1}{\mu} \nabla \times \mathbf{E}$ by $\varepsilon k_T^2 \mathbf{E}$ and sum over Trefftz elements:

The symmetric DG Trefftz formulation

A Green formula on one element T:

$$\int_T (\nabla \times \frac{1}{\mu_T} \nabla \times \mathbf{E}) \cdot \boldsymbol{\omega} dx = \int_T (\frac{1}{\mu_T} \nabla \times \mathbf{E}) \cdot (\nabla \times \boldsymbol{\omega}) dx + \int_{\partial T} \mathbf{n} \times (\frac{1}{\mu_T} \nabla \times \mathbf{E})|_{\partial \Omega} \cdot \boldsymbol{\omega} ds$$

Replace $\nabla \times \frac{1}{\mu} \nabla \times \mathbf{E}$ by $\varepsilon k_T^2 \mathbf{E}$ and sum over Trefftz elements:

$$\sum_{T \in \mathcal{T}} \int_T \left(\frac{1}{\mu} (\nabla \times \mathbf{E}) \cdot (\nabla \times \boldsymbol{\omega}) - \varepsilon k_T^2 \mathbf{E} \cdot \boldsymbol{\omega} \right) dx = \sum_{T \in \mathcal{T}} \int_{\partial T} (\frac{1}{\mu} \nabla \times \mathbf{E})|_{\partial T} \cdot \mathbf{n}_T \times \boldsymbol{\omega}_T ds$$

The symmetric DG Trefftz formulation

A Green formula on one element T:

$$\int_T (\nabla \times \frac{1}{\mu_T} \nabla \times \mathbf{E}) \cdot \boldsymbol{\omega} dx = \int_T (\frac{1}{\mu_T} \nabla \times \mathbf{E}) \cdot (\nabla \times \boldsymbol{\omega}) dx + \int_{\partial T} \mathbf{n} \times (\frac{1}{\mu_T} \nabla \times \mathbf{E})|_{\partial\Omega} \cdot \boldsymbol{\omega} ds$$

Replace $\nabla \times \frac{1}{\mu} \nabla \times \mathbf{E}$ by $\varepsilon k_T^2 \mathbf{E}$ and sum over Trefftz elements:

$$\sum_{T \in \mathcal{T}} \int_T \left(\frac{1}{\mu} (\nabla \times \mathbf{E}) \cdot (\nabla \times \boldsymbol{\omega}) - \varepsilon k_T^2 \mathbf{E} \cdot \boldsymbol{\omega} \right) dx = \sum_{T \in \mathcal{T}} \int_{\partial T} (\frac{1}{\mu} \nabla \times \mathbf{E})|_{\partial T} \cdot \mathbf{n}_T \times \boldsymbol{\omega}_T ds$$

The symmetric DG Trefftz formulation

We can invert the roles of E and ω because ω verifies Maxwell on each element:

$$\sum_{T\in\mathcal{T}}\int_T \left(\frac{1}{\mu}(\nabla\times\mathbf{E})\cdot(\nabla\times\boldsymbol{\omega})-\varepsilon k_T^2\mathbf{E}\cdot\boldsymbol{\omega}\right)dx=\sum_{T\in\mathcal{T}}\int_{\partial T}(\frac{1}{\mu}\nabla\times\boldsymbol{\omega})|_{\partial T}\cdot\mathbf{n}_T\times\mathbf{E}_T ds$$

Compute difference:

$$\sum_{T\in\mathcal{T}}\int_{\partial T}\left((\frac{1}{\mu}\nabla\times\mathbf{E})|_{\partial T}\cdot\mathbf{n}_T\times\boldsymbol{\omega}_T-(\frac{1}{\mu}\nabla\times\boldsymbol{\omega})|_{\partial T}\cdot\mathbf{n}_T\times\mathbf{E}_T\right)ds=0.$$

The symmetric DG Trefftz formulation

Notation: sums and differences on edges

On an edge $F = \partial T \cap \partial K$ we define

The trace jump: $$[[\omega]] = \mathbf{n}_T \times \boldsymbol{\omega}_T + \mathbf{n}_K \times \boldsymbol{\omega}_K .$$

The trace average: $$\{\{\omega\}\} = \frac{1}{2}(\boldsymbol{\omega}_T + \boldsymbol{\omega}_K) .$$

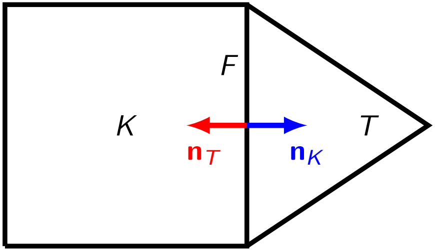


Figure: Schematic view over two neighbouring Trefftz elements sharing a common edge F with normals.

The symmetric DG Trefftz formulation

The variational formulation

$$\begin{cases} \mathbf{E} \in X_{\mathcal{T}}, \forall \boldsymbol{\omega} \in X_{\mathcal{T}} \\ a(\mathbf{E}, \boldsymbol{\omega}) = L\boldsymbol{\omega} \end{cases}$$

The symmetric DG Trefftz formulation

The variational formulation

$$\begin{cases} \mathbf{E} \in X_{\mathcal{T}}, \forall \boldsymbol{\omega} \in X_{\mathcal{T}} \\ \\ a(\mathbf{E}, \boldsymbol{\omega}) = L\boldsymbol{\omega} \end{cases}$$

$$\int_{\Gamma} \{\{\mathbf{E}\}\} \cdot [[\frac{1}{\mu}\nabla \times \boldsymbol{\omega}]] + [[\frac{1}{\mu}\nabla \times \mathbf{E}]] \cdot \{\{\boldsymbol{\omega}\}\}$$
$$-\{\{\tfrac{1}{\mu}\nabla \times \mathbf{E}\}\} \cdot [[\boldsymbol{\omega}]] - [[\mathbf{E}]] \cdot \{\{\tfrac{1}{\mu}\nabla \times \boldsymbol{\omega}\}\} ds$$
$$+ \int_{\partial\Omega} \mathbf{E} \cdot \mathbf{n} \times (\frac{1}{\mu}\nabla \times \boldsymbol{\omega}) + \mathbf{n} \times (\frac{1}{\mu}\nabla \times \mathbf{E}) \cdot \boldsymbol{\omega} ds$$
$$-2 \int_{\partial\Omega_R} \tfrac{1}{Z}(\tfrac{1}{\mu}\nabla \times \mathbf{E}) \cdot (\tfrac{1}{\mu}\nabla \times \boldsymbol{\omega}) ds$$

The symmetric DG Trefftz formulation

The variational formulation

$$\begin{cases} \mathbf{E} \in X_{\mathcal{T}}, \forall \boldsymbol{\omega} \in X_{\mathcal{T}} \\ a(\mathbf{E}, \boldsymbol{\omega}) = L\boldsymbol{\omega} \end{cases}$$

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$$-\{\{\tfrac{1}{\mu}\nabla \times \mathbf{E}\}\} \cdot [[\boldsymbol{\omega}]] - [[\mathbf{E}]] \cdot \{\{\tfrac{1}{\mu}\nabla \times \boldsymbol{\omega}\}\} ds$$
$$+ \int_{\partial\Omega} \mathbf{E} \cdot \mathbf{n} \times (\frac{1}{\mu}\nabla \times \boldsymbol{\omega}) + \mathbf{n} \times (\frac{1}{\mu}\nabla \times \mathbf{E}) \cdot \boldsymbol{\omega} ds$$
$$-2 \int_{\partial\Omega_R} \tfrac{1}{Z}(\tfrac{1}{\mu}\nabla \times \mathbf{E})\dot{(}\tfrac{1}{\mu}\nabla \times \boldsymbol{\omega})ds$$
$$= -2 \int_{\partial\Omega_R} \frac{1}{Z}\mathbf{g_R} \cdot (\frac{1}{\mu}\nabla \times \boldsymbol{\omega})ds + 2\int_{\partial\Omega_D} \mathbf{g_D} \cdot \mathbf{n} \times (\frac{1}{\mu}\nabla \times \boldsymbol{\omega})ds.$$

The symmetric DG Trefftz formulation

The local problem: an isomorphism

$$\begin{cases} \nabla \times \dfrac{1}{\mu_T} \nabla \times \boldsymbol{\omega}_g - \varepsilon_T k_T^2 \boldsymbol{\omega}_g = 0 & \text{in} \quad T \\ \mathbf{n} \times (\dfrac{1}{\mu_T} \nabla \times \boldsymbol{\omega}_g) - Z_T \mathbf{n} \times (\boldsymbol{\omega}_g \times \mathbf{n}) = \mathbf{g} & \text{on} \quad \partial T \end{cases}$$

Solved with impedance BC

$$\begin{aligned} \mathcal{L} : L^2(\partial T) &\to X_T \\ \mathbf{g} &\mapsto \mathcal{L}(\mathbf{g}) = \boldsymbol{\omega}_g \end{aligned}$$

The symmetric DG Trefftz formulation

The local problem: an isomorphism

$$\begin{cases} \nabla \times \dfrac{1}{\mu_T} \nabla \times \boldsymbol{\omega}_g - \varepsilon_T k_T^2 \boldsymbol{\omega}_g = 0 & \text{in} \quad T \\ \mathbf{n} \times (\dfrac{1}{\mu_T} \nabla \times \boldsymbol{\omega}_g) - Z_T \mathbf{n} \times (\boldsymbol{\omega}_g \times \mathbf{n}) = \mathbf{g} & \text{on} \quad \partial T \end{cases}$$

$$\mathcal{L} : L^2(\partial T) \to X_T$$
$$\mathbf{g} \mapsto \mathcal{L}(\mathbf{g}) = \boldsymbol{\omega}_g$$

Solved with impedance BC

$Im(Z_T) > 0$, example:
$Z_T = i\sqrt{\mu_T \varepsilon_T} k$

The symmetric DG Trefftz formulation

The local problem: an isomorphism

$$\begin{cases} \nabla \times \dfrac{1}{\mu_T} \nabla \times \boldsymbol{\omega}_g - \varepsilon_T k_T^2 \boldsymbol{\omega}_g = 0 & \text{in} \quad T \\[2ex] \mathbf{n} \times (\dfrac{1}{\mu_T} \nabla \times \boldsymbol{\omega}_g) - Z_T \mathbf{n} \times (\boldsymbol{\omega}_g \times \mathbf{n}) = \mathbf{g} & \text{on} \quad \partial T \end{cases}$$

$$\mathcal{L} : L^2(\partial T) \to X_{\mathcal{T}}$$
$$\mathbf{g} \mapsto \mathcal{L}(\mathbf{g}) = \boldsymbol{\omega}_g$$

Solved with impedance BC

$Im(Z_T) > 0$, example:

$Z_T = i\sqrt{\mu_T \varepsilon_T} k$

Well posed.

The symmetric DG Trefftz formulation

The local problem: an isomorphism

$$\begin{cases} \nabla \times \dfrac{1}{\mu_T} \nabla \times \boldsymbol{\omega}_g - \varepsilon_T k_T^2 \boldsymbol{\omega}_g = 0 & \text{in} \quad T \\ \mathbf{n} \times (\dfrac{1}{\mu_T} \nabla \times \boldsymbol{\omega}_g) - Z_T \mathbf{n} \times (\boldsymbol{\omega}_g \times \mathbf{n}) = \mathbf{g} & \text{on} \quad \partial T \end{cases}$$

$$\mathcal{L} : L^2(\partial T) \to X_T$$
$$\mathbf{g} \mapsto \mathcal{L}(\mathbf{g}) = \boldsymbol{\omega}_g$$

Solved with impedance BC

$Im(Z_T) > 0$, example:
$Z_T = i\sqrt{\mu_T \varepsilon_T} k$

Well posed.

Direct acces to curl trace.

The symmetric DG Trefftz formulation

$$
\begin{tikzcd}
L^2(\partial T) \arrow[r, bend right, "\mathcal{L}"'] & X_T \arrow[l, bend right, "\mathcal{L}^{-1}"'] \arrow[dr, "\subset"] & \\
\mathcal{P}_h^q(\partial T) \arrow[u, "\subset"] \arrow[r, "\mathcal{L}"] \arrow[dr, "\mathcal{L}_h^p"] & X_h^q(T) \arrow[u, "\subset"] \arrow[r, "\subset"] & H(curl, T) \\
& \mathcal{L}_h^p(\mathcal{P}_h^q(\partial T)) \arrow[r, "\subset"] & \mathcal{N}_h^p(T) \arrow[u, "\subset"]
\end{tikzcd}
$$

Figure: Relationship between all relevant function spaces. Arrows with $\subset$ denote inclusion maps.

The symmetric DG Trefftz formulation

$$
\begin{tikzcd}
L^2(\partial T) \arrow[r, bend right, "\mathcal{L}"'] & X_T \arrow[l, bend right, "\mathcal{L}^{-1}"'] \arrow[dr, "\subset"] & \\
\mathcal{P}_h^q(\partial T) \arrow[u, "\subset"] \arrow[r, "\mathcal{L}"] \arrow[dr, "\mathcal{L}_h^p"'] & X_h^q(T) \arrow[u, "\subset"] \arrow[r, "\subset"] & H(curl, T) \\
& \mathcal{L}_h^p(\mathcal{P}_h^q(\partial T)) \arrow[r, "\subset"] & \mathcal{N}_h^p(T) \arrow[u, "\subset"]
\end{tikzcd}
$$

Figure: Relationship between all relevant function spaces. Arrows with $\subset$ denote inclusion maps.

Approximation of impedance boundary condition:

$$\mathcal{P}_h^q(\partial T) := \{\mathbf{g} \in L^2(\partial T), \quad \mathbf{g}|_f \in \mathcal{P}^q(f), \quad \forall f, \quad f \in \mathcal{F}_{\partial T}^h\}$$

The symmetric DG Trefftz formulation

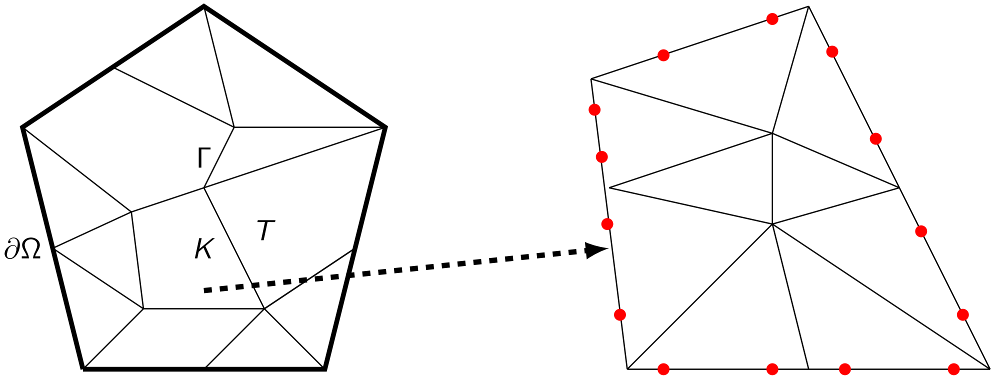


Figure: To the left: the Trefftz DG mesh. Γ is the interior edges. K and T are two neighbouring Trefftz elements. To the right: one Trefftz element with a triangular FE mesh.

Approximation of the local problem: Nedelec finite element method

The weak formulation:

$$\begin{cases} \mathbf{E} \in H_0(curl, \Omega), \forall \boldsymbol{\omega} \in H_0(curl, \Omega) \\ \displaystyle\int_\Omega (\frac{1}{\mu} \nabla \times \mathbf{E}) \cdot (\nabla \times \boldsymbol{\omega}) dx - k^2 \int_\Omega \varepsilon \mathbf{E} \cdot \boldsymbol{\omega} dx \\ \qquad\qquad\qquad + Z \int_{\partial\Omega_R} \mathbf{n} \times (\mathbf{E} \times \mathbf{n}) \cdot \boldsymbol{\omega} ds = - \int_{\partial\Omega_R} \mathbf{g} \cdot \boldsymbol{\omega} ds \end{cases}$$

Inhomogenous Dirichlet data is treated with a variable change.

Approximation of the local problem: Nedelec finite element method

Validation of method

$$\mathbf{k} = k \begin{bmatrix} cos(\theta) \\ sin(\theta) \end{bmatrix}, \quad \mathbf{E} = E \begin{bmatrix} -sin(\theta) \\ cos(\theta) \end{bmatrix} e^{-i\mathbf{kx}}$$

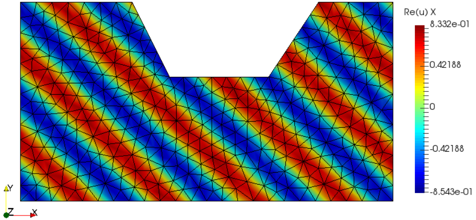


Figure: A plane wave test: $\theta = 45$ degrees.
$p = 1, h = 0.13, k = 4\pi$.

Approximation of the local problem: Nedelec finite element method

Validation of method

$$\mathbf{k} = k \begin{bmatrix} cos(\theta) \\ sin(\theta) \end{bmatrix}, \quad \mathbf{E} = E \begin{bmatrix} -sin(\theta) \\ cos(\theta) \end{bmatrix} e^{-i\mathbf{kx}}$$

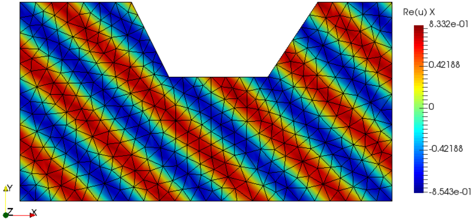


Figure: A plane wave test: $\theta = 45$ degrees. $p = 1$, $h = 0.13$, $k = 4\pi$.

$$||.||_{H^1(curl)} = \sqrt{\int_\Omega |.|^2 dx + \int_\Omega \frac{1}{k^2} |\nabla \times .|^2 dx}$$

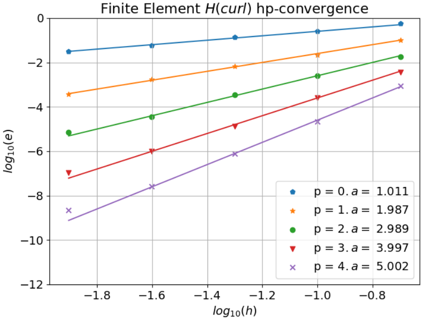
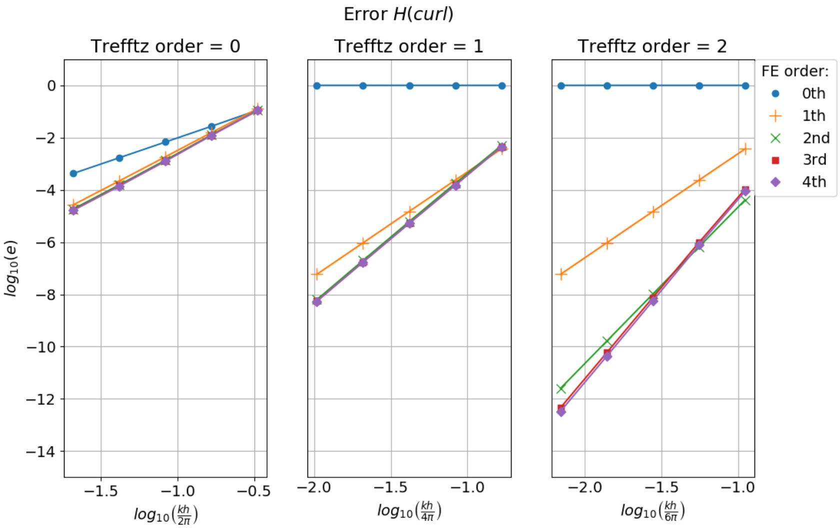


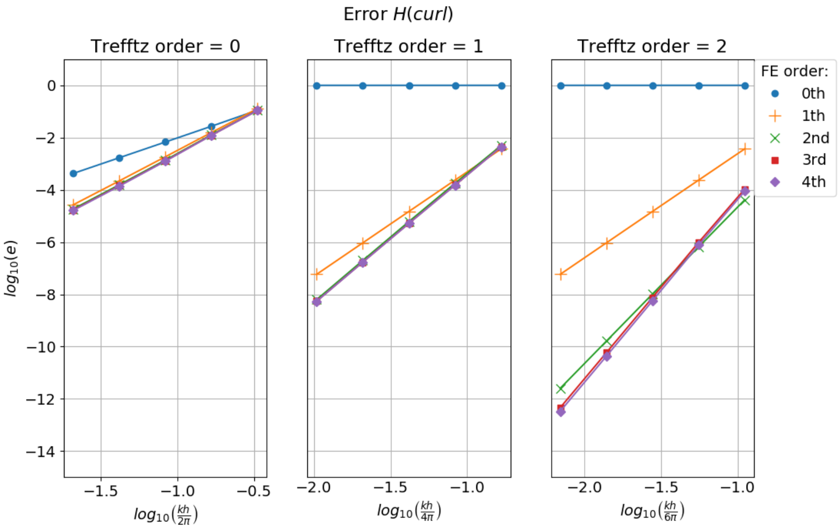
Figure: Relative error convergence rate: $e = Ch^{p+1}$

Approximation of the local problem: Nedelec finite element method



Rule of thumb: $p \geq q$

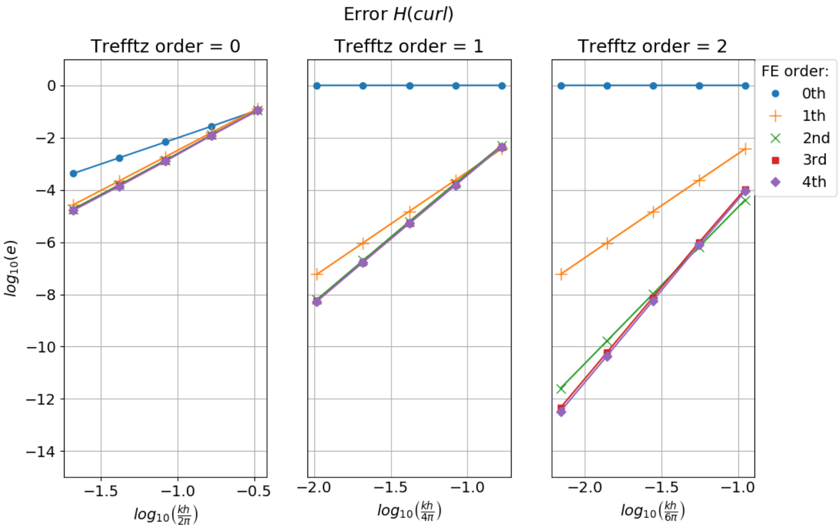
Approximation of the local problem: Nedelec finite element method



Rule of thumb: $p \geq q$

$p < q \rightarrow$ precision loss

Approximation of the local problem: Nedelec finite element method

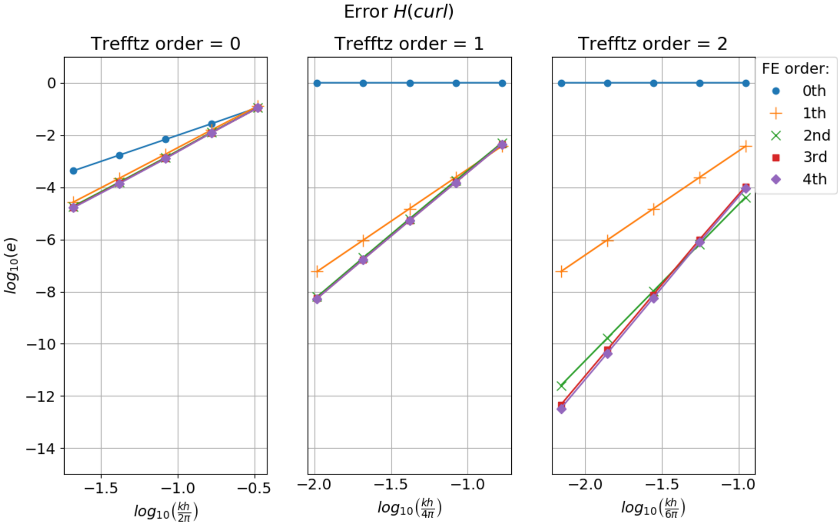


Rule of thumb: $p \geq q$

$p < q \rightarrow$ precision loss

Super convergence

Approximation of the local problem: Nedelec finite element method



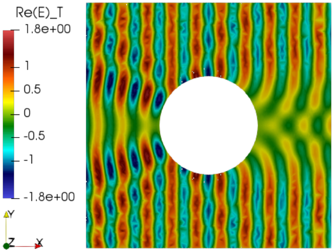
Rule of thumb: $p \geq q$

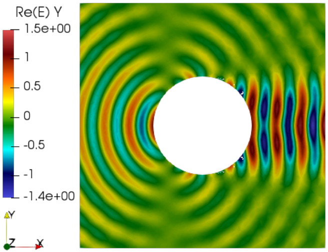
$p < q \rightarrow$ precision loss

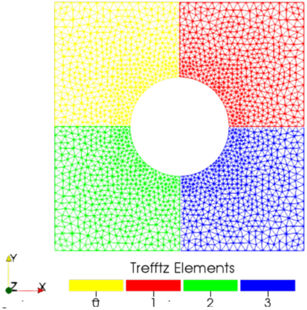
Super convergence

Increasing p does not change the Trefftz DOFs

A qualitative result: scattering on a perfect conducting circle







$\mathbf{E}_{total} = \mathbf{E}_{incident} + \mathbf{E}_{scatter}$

Incident plane wave from the left

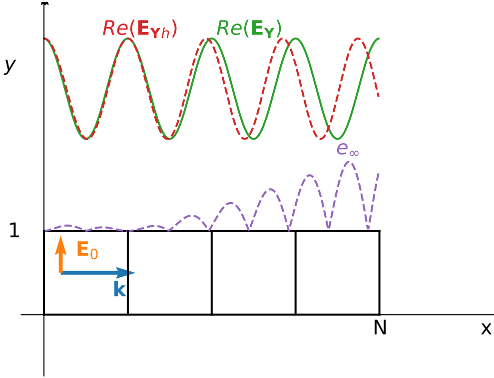
On the left: $\mathbf{E}_{total}$

On the right: $\mathbf{E}_{scatter}$

Long range propagation: Numerical dispersion

$$
\begin{cases}
\nabla \times \nabla \times \mathbf{E} - k^2\mathbf{E} = 0 \quad in \quad \Omega = [0, L] \times [0, 1] \\
\mathbf{n} \times (\mathbf{E} \times \mathbf{n}) = 0, \quad y \in \{0, 1\} \\
\mathbf{n} \times (\mathbf{E} \times \mathbf{n}) = -e_y, \quad x = 0 \\
\mathbf{n} \times (\frac{1}{\mu}\nabla \times \mathbf{E}) - ik\mathbf{n} \times (\mathbf{E} \times \mathbf{n}) = 0, \quad x = L
\end{cases}
$$

$$
e_\infty = 100 \max_{(x,y)\in\Omega} \frac{|\mathbf{E} - \mathbf{E}_h|}{||\mathbf{E}||_\infty}.
$$



Long range propagation: Numerical dispersion

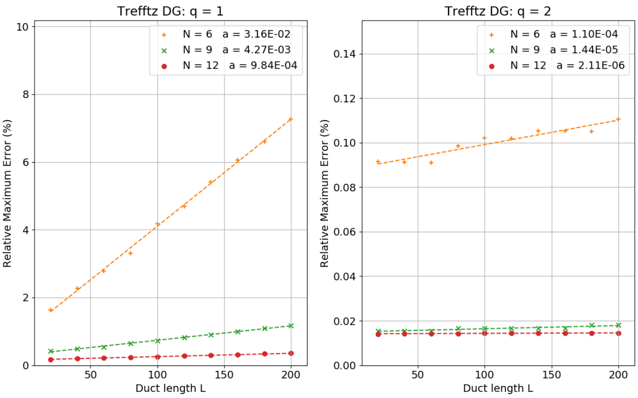


Figure: Relative maximum order for long range propagation using the FE method for different orders and meshes. N on this figure corresponds to $1/h$.

Long range propagation: Numerical dispersion

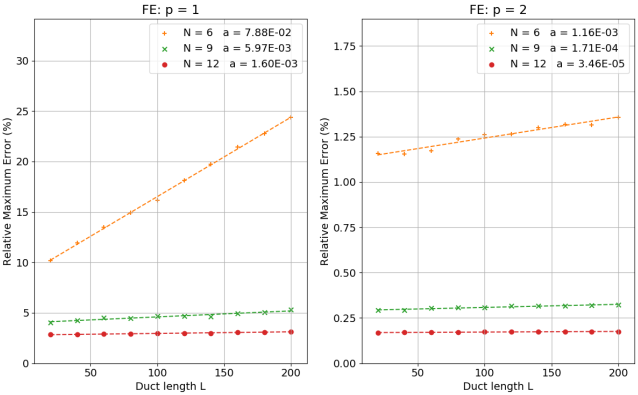
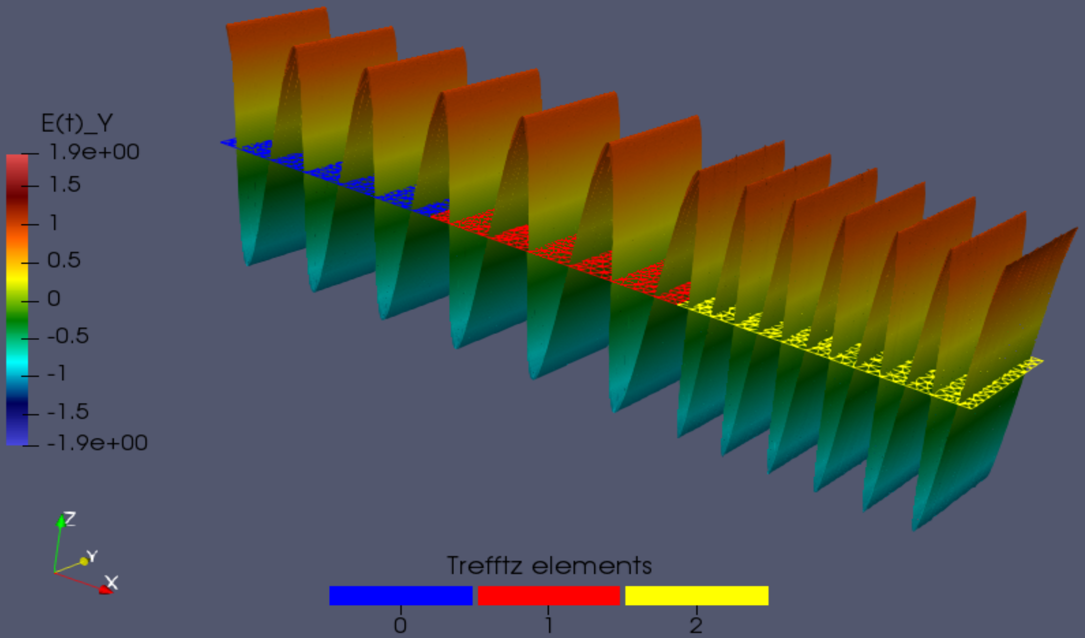
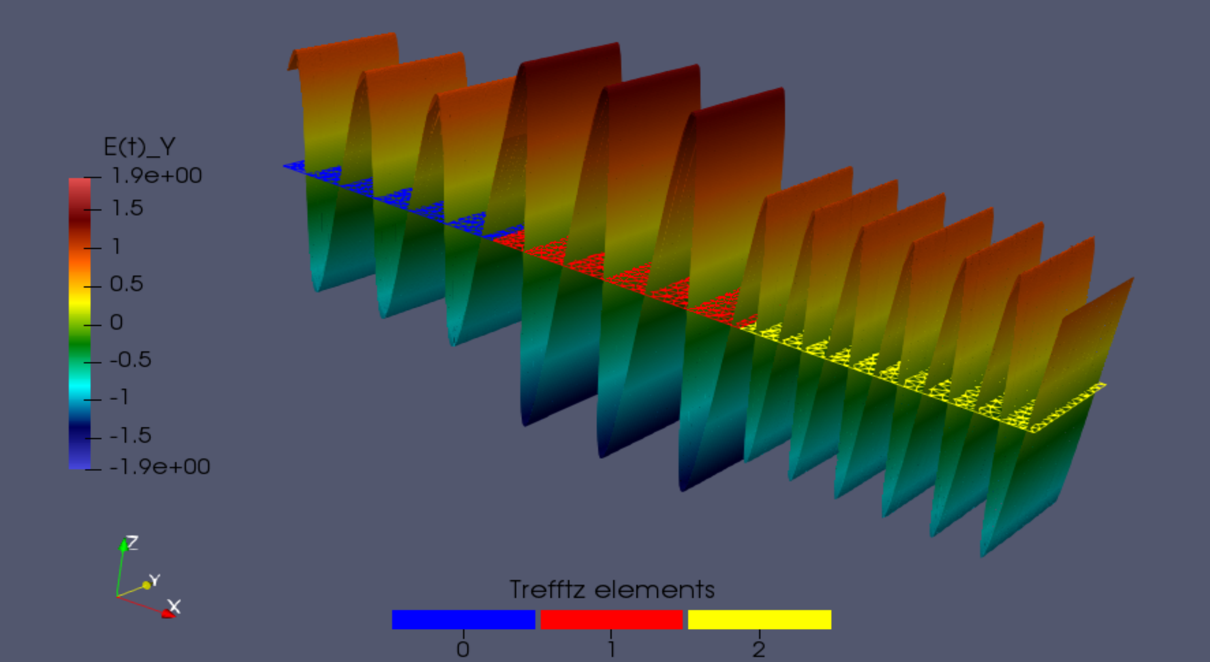
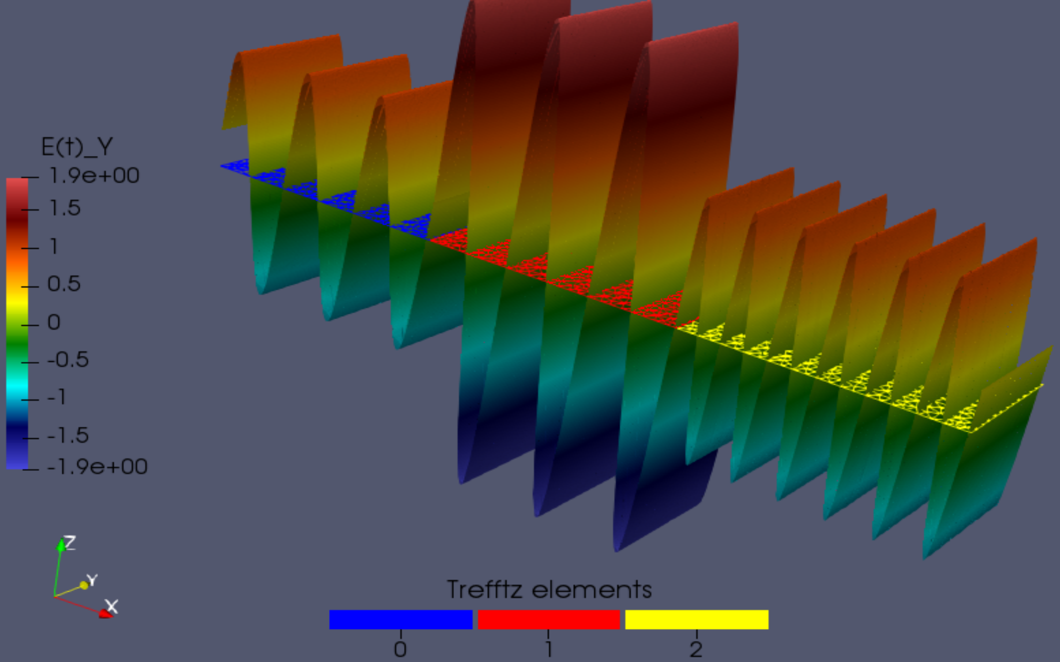


Figure: Relative maximum order for long range propagation using the FE method for different orders and meshes.


E(t)_Y
1.9e+00
1.5
1
0.5
0
-0.5
-1
-1.5
-1.9e+00
Z
Y
X
Trefftz elements
0
1
2


E(t)_Y
1.9e+00
1.5
1
0.5
0
-0.5
-1
-1.5
-1.9e+00
Z
Y
X
Trefftz elements
0
1
2


E(t)_Y
1.9e+00
1.5
1
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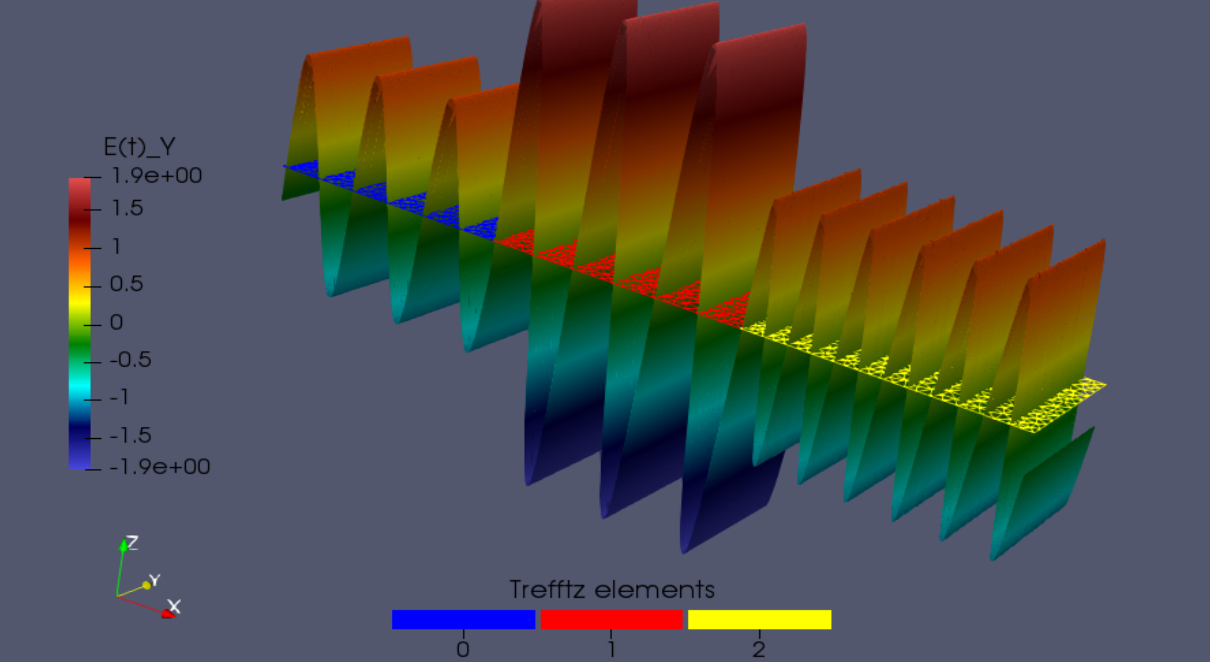
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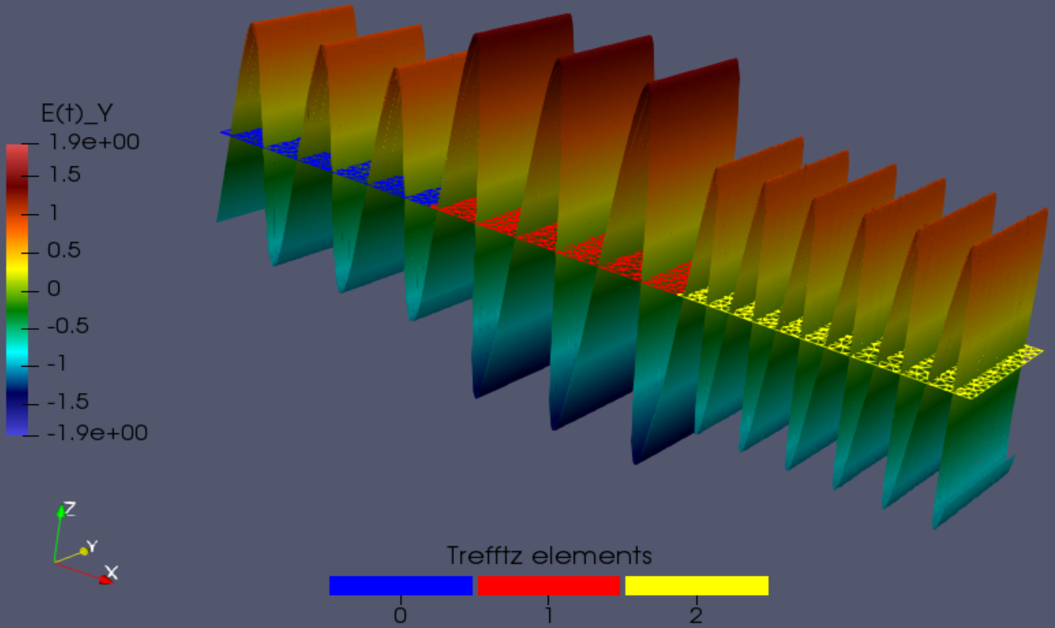
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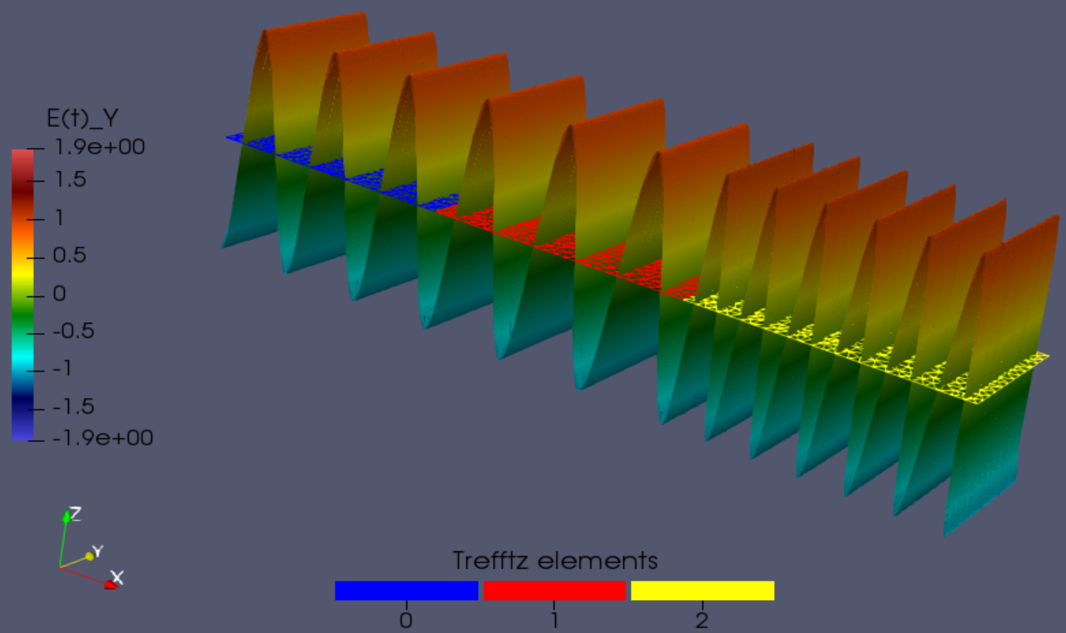
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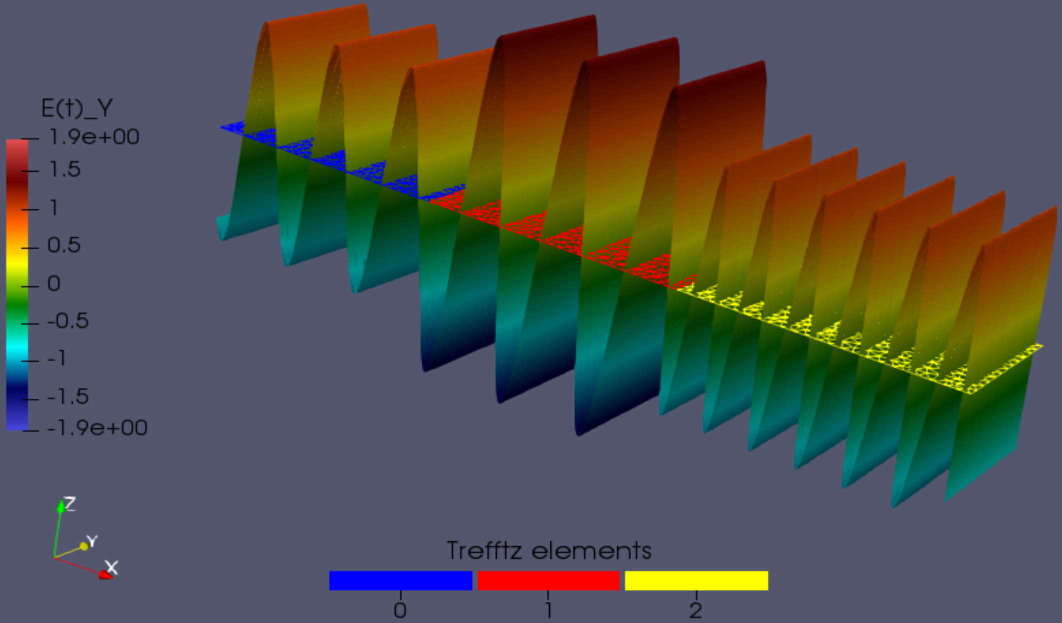
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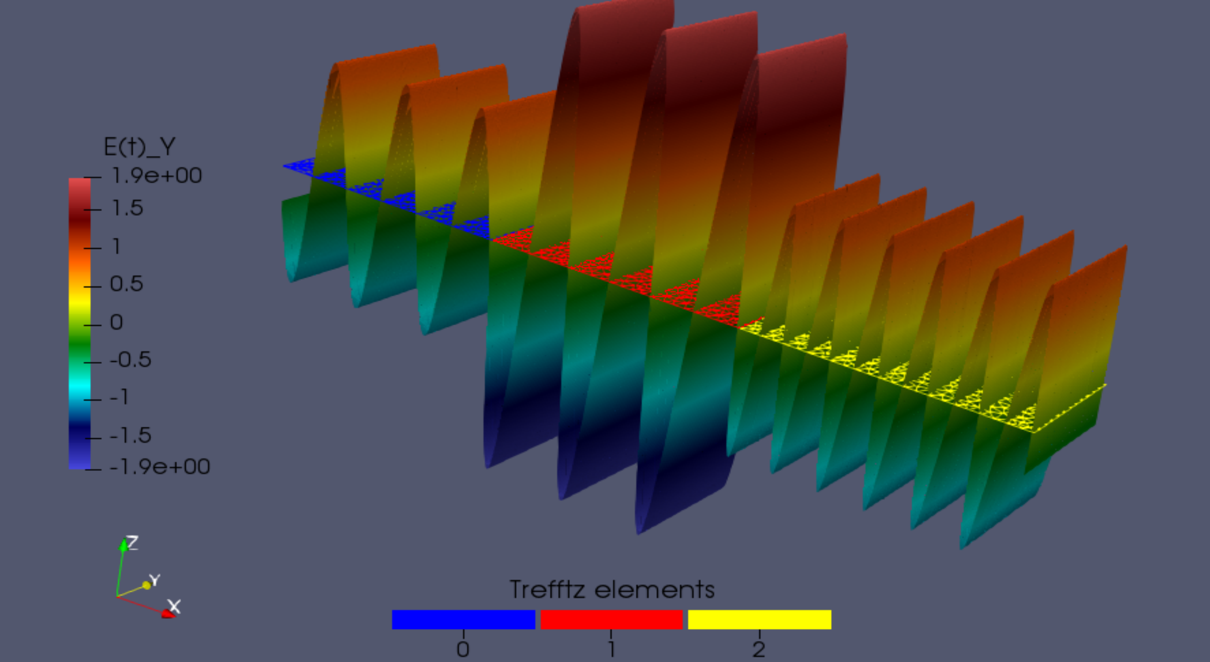
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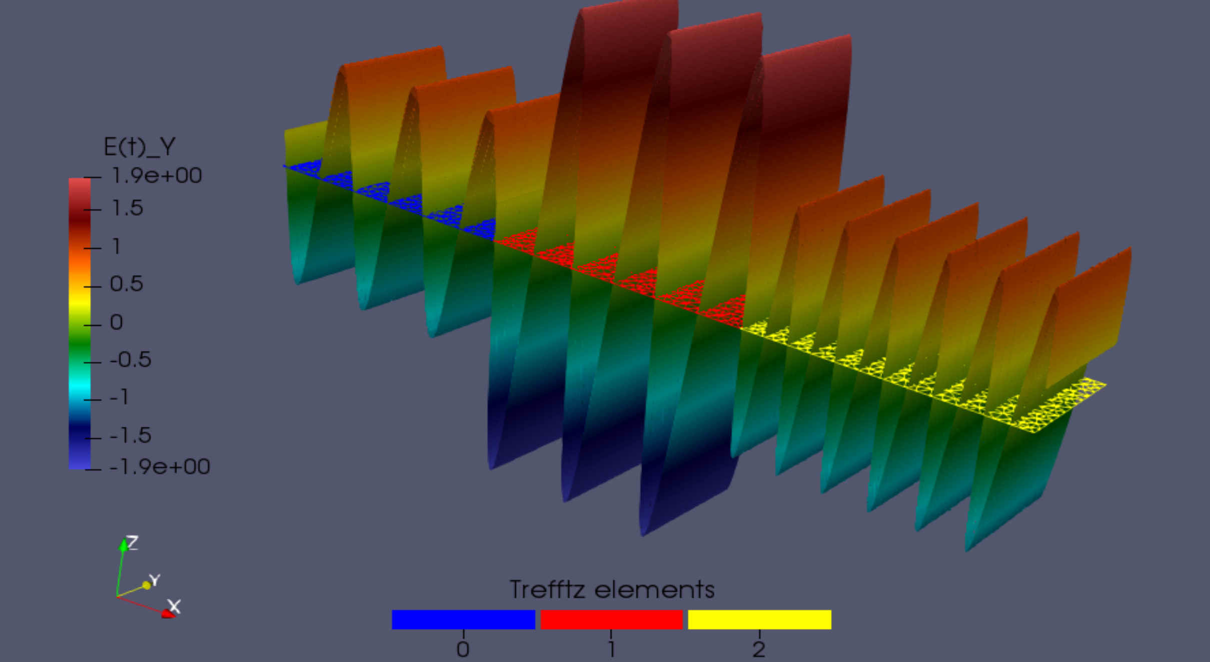
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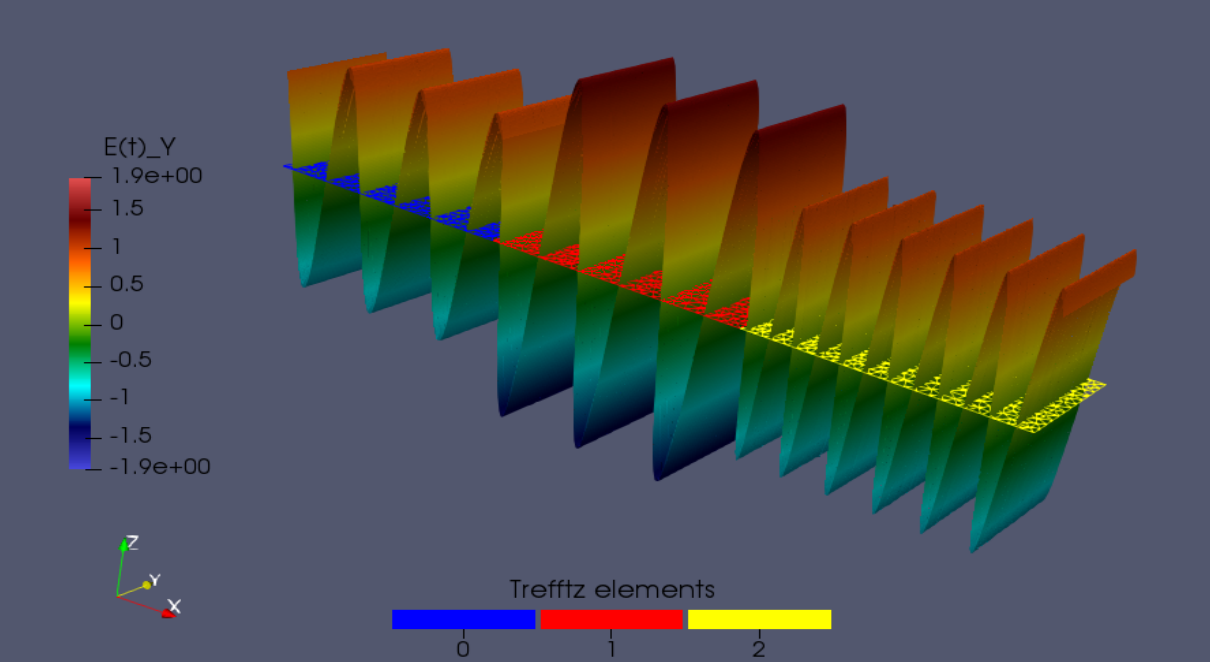
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Trefftz elements
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Conclusion and perspectives

- Trefftz IPDG formulation combined with BEM reduces the pollution effect
- FEM UWVF shows the same properties. We have only very preliminary results that can not be yet presented
- The future: extension to elastic waves. Not that obvious when considering BEM or FEM...

With Sébastien Pernet, we are planning to answer to call to get a PhD thesis.