Schémas Bas-Froude pour le modèle Shallow Water multicouches

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INSTITUT de MATHEMATIQUES de TOULOUSE

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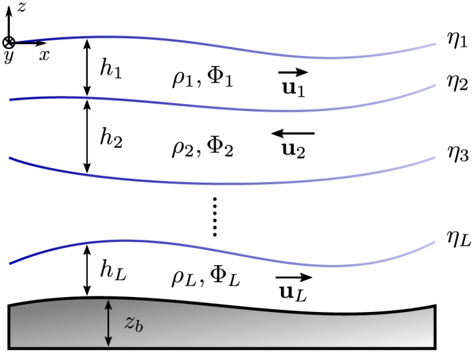
Présentation du modèle

Shallow Water multicouches

$$\begin{aligned} &\partial_t h_i + div(h_i \mathbf{u}_i) = 0\,, \\ &\partial_t (h_i \mathbf{u}_i) + div(h_i \mathbf{u}_i \otimes \mathbf{u}_i) + h_i \nabla \phi_i = 0\,. \end{aligned} \qquad i = 1, \cdots, L\,.$$

- ▷ hauteur d'eau h_i
- ▷ vitesse $\mathbf{u}_i = (u_i, v_i)$
- ▷ densité ρ_i
- ▷ pression

$$\phi_i = g \sum_{j=1}^{L} \frac{\rho_j h_j}{\rho_{max(i,j)}}$$



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Difficultés :

Structure

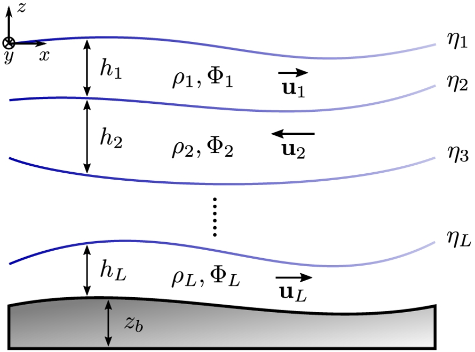
- ▷ Non conservatif
- ▷ Hyperbolicité

Cadre applicatif

- ▷ Bas-Froude

Stabilité

- ▷ Energie
- ▷ Robustesse
- ▷ Etats d'équilibre



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▷ Energie potentielle $\mathcal{E}$: $\partial_{\rho_i h_i} \mathcal{E} = \phi_i = g \sum_{j=1}^{L} \dfrac{\rho_j h_j}{\rho_{max(i,j)}}$.

▷ Energie cinétique : $\mathcal{K}_i = \dfrac{1}{2} \rho_i h_i \|\mathbf{u}_i\|^2$.

▷ Energie totale : $E = \mathcal{E} + \sum_{i=1}^{L} \mathcal{K}_i$.

Bilans d'énergie (solutions régulières)

$$\partial_t \mathcal{E} + \sum_{i=1}^{L} div\left(\rho_i h_i \phi_i \mathbf{u}_i\right) - \sum_{i=1}^{L} \rho_i h_i \mathbf{u}_i . \nabla \phi_i = 0\,,$$

$$\partial_t \mathcal{K}_i + div\left(\mathcal{K}_i \mathbf{u}_i\right) + \rho_i h_i \mathbf{u}_i . \nabla \phi_i = 0\,,$$

$$\partial_t E + \sum_{i=1}^{L} div\left[\left(\rho_i h_i \phi_i + \mathcal{K}_i\right) \mathbf{u}_i\right] = 0.$$

Equations régularisées

Shallow Water multicouches régularisé

$$\begin{aligned} &\partial_t h_i + div(h_i(\mathbf{u}_i - \delta\mathbf{u}_i)) = 0\,, \\ &\partial_t(h_i\mathbf{u}_i) + div(h_i\mathbf{u}_i \otimes (\mathbf{u}_i - \delta\mathbf{u}_i)) + h_i\nabla\phi_i = 0\,. \end{aligned} \qquad i = 1, \cdots, L\,.$$

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Bilans d'énergie (solutions régulières)

$$\partial_t\mathcal{E} + \sum_{i=1}^{L} div\left(\rho_i h_i\phi_i\mathbf{u}_i\right) - \sum_{i=1}^{L}\rho_i h_i\mathbf{u}_i.\nabla\phi_i = 0\,,$$

$$\partial_t\mathcal{K}_i + div\left(\mathcal{K}_i\mathbf{u}_i\right) + \rho_i h_i\mathbf{u}_i.\nabla\phi_i = 0\,,$$

$$\partial_t E + \sum_{i=1}^{L} div\left[\left(\rho_i h_i\phi_i + \mathcal{K}_i\right)\mathbf{u}_i\right] = 0.$$

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Shallow Water multicouches régularisé

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Bilans d'énergie (solutions régulières)

$$\partial_t\mathcal{E} + \sum_{i=1}^{L} div\left(\rho_i h_i\phi_i(\mathbf{u}_i - \delta\mathbf{u}_i)\right) - \sum_{i=1}^{L}\rho_i h_i\mathbf{u}_i.\nabla\phi_i = -\sum_{i=1}^{L}\rho_i h_i\delta\mathbf{u}_i.\nabla\phi_i\,,$$
$$\partial_t\mathcal{K}_i + div\left(\mathcal{K}_i(\mathbf{u}_i - \delta\mathbf{u}_i)\right) + \rho_i h_i\mathbf{u}_i.\nabla\phi_i = 0\,,$$
$$\partial_t E + \sum_{i=1}^{L} div\left[\left(\rho_i h_i\phi_i + \mathcal{K}_i\right)(\mathbf{u}_i - \delta\mathbf{u}_i)\right] = -\sum_{i=1}^{L}\rho_i h_i\delta\mathbf{u}_i.\nabla\phi_i.$$

Equations régularisées

Shallow Water multicouches régularisé

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Bilans d'énergie (solutions régulières)

$$\partial_t\mathcal{E} + \sum_{i=1}^{L} div\,(\rho_i h_i\phi_i(\mathbf{u}_i - \delta\mathbf{u}_i)) - \sum_{i=1}^{L}\rho_i h_i\mathbf{u}_i.\nabla\phi_i = -\sum_{i=1}^{L}\rho_i h_i\delta\mathbf{u}_i.\nabla\phi_i\,,$$

$$\partial_t\mathcal{K}_i + div\,(\mathcal{K}_i(\mathbf{u}_i - \delta\mathbf{u}_i)) + \rho_i h_i\mathbf{u}_i.\nabla\phi_i = 0\,,$$

$$\partial_t E + \sum_{i=1}^{L} div\left[\left(\rho_i h_i\phi_i + \mathcal{K}_i\right)(\mathbf{u}_i - \delta\mathbf{u}_i)\right] = -\sum_{i=1}^{L}\rho_i h_i\delta\mathbf{u}_i.\nabla\phi_i.$$

$\triangleright\ \delta\mathbf{u}_i = \gamma\nabla\phi_i$: second membre **régularisant.**

Modèle augmenté

▷ On introduit la vorticité $\omega = \partial_x v - \partial_y u$ comme variable dans le système.

Modèle augmenté régularisé

$$\partial_t h_i + div(h_i(\mathbf{u}_i - \delta\mathbf{u}_i)) = 0\,,$$
$$\partial_t(h_i\mathbf{u}_i) + div(h_i\mathbf{u}_i \otimes (\mathbf{u}_i - \delta\mathbf{u}_i)) + h_i\nabla\phi_i = 0\,.$$

▷ Energie totale : $E = \mathcal{E} + \sum_{i=1}^{L} \mathcal{K}_i$

Bilan d'énergie (solutions continues)

$$\partial_t E + \sum_{i=1}^{L} div\Big[\Big(h_i\phi_i + \mathcal{K}_i\Big)(\mathbf{u}_i - \delta\mathbf{u}_i)\Big] = -\sum_{i=1}^{L} \rho_i h_i \delta\mathbf{u}_i.\nabla\phi_i\,,$$

Modèle augmenté

- ▷ On introduit la vorticité $\omega = \partial_x v - \partial_y u$ comme variable dans le système.
- ▷ On pose : $q_i = \dfrac{\omega_i}{h_i}$. Le système multicouches devient :

Modèle augmenté régularisé

$$\begin{aligned}
&\partial_t h_i + div(h_i(\mathbf{u}_i - \delta\mathbf{u}_i)) = 0\,,\\
&\partial_t(h_i\mathbf{u}_i) + div(h_i\mathbf{u}_i \otimes (\mathbf{u}_i - \delta\mathbf{u}_i)) + h_i\nabla\phi_i = 0\,,\\
&\partial_t(h_i q_i) + div(h_i q_i(\mathbf{u}_i - \delta\mathbf{u}_i)) = 0\,.
\end{aligned}$$

- ▷ Energie associée à la vorticité $\mathcal{W}_i = \dfrac{1}{2}\rho_i h_i {q_i}^2$.
- ▷ Energie totale : $E_a = E + \sum_{i=1}^{L}\mathcal{W}_i = \mathcal{E} + \sum_{i=1}^{L}\mathcal{K}_i + \sum_{i=1}^{L}\mathcal{W}_i$.

Bilan d'énergie (solutions continues)

$$\partial_t E_a + \sum_{i=1}^{L} div\left[\Big(h_i\phi_i + \mathcal{K}_i + \mathcal{W}_i\Big)(\mathbf{u}_i - \delta\mathbf{u}_i)\right] = -\sum_{i=1}^{L}\rho_i h_i \delta\mathbf{u}_i.\nabla\phi_i\,,$$

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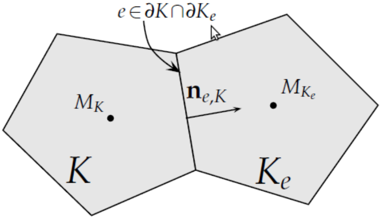
Schéma numérique (Parisot - Vila , 2015)

Schéma numérique

$$h_{K,i}^{n+1} = h_{K,i}^{n} - \frac{\Delta t}{m_K} \sum_{e\in\partial K} \mathcal{F}_{e,i}^{n+1}.\mathbf{n}_{e,K}\, m_e \,,$$

$$h_{K,i}^{n+1}\mathbf{u}_{K,i}^{n+1} = h_{K,i}^{n}\mathbf{u}_{K,i}^{n} - \frac{\Delta t}{m_K} \sum_{e\in\partial K} \left(\mathbf{u}_{K,i}^{n} \left(\mathcal{F}_{e,i}^{n+1}.\mathbf{n}_{e,K}\right)^{+} - \mathbf{u}_{K_e,i}^{n} \left(\mathcal{F}_{e,i}^{n+1}.\mathbf{n}_{e,K}\right)^{-} \right) m_e$$
$$- \frac{\Delta t}{m_K} h_{K,i}^{n+1} \sum_{e\in\partial K} \Phi_{e,i}^{n+1} m_e \,,$$

$$m_K = |K| \quad ; \quad m_e = |e| \,,$$
$$\omega^{\pm} = \frac{|\omega| \pm \omega}{2} \,,$$
$$\Phi_{e,i}^{n+1} = \frac{\Phi_{K,i}^{n+1} + \Phi_{K_e,i}^{n+1}}{2} \,.$$



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$$h_{K,i}^{n+1} = h_{K,i}^{n} - \frac{\Delta t}{m_K} \sum_{e \in \partial K} \mathcal{F}_{e,i}^{n+1} . \mathbf{n}_{e,K} \, m_e \, ,$$

$$h_{K,i}^{n+1} \mathbf{u}_{K,i}^{n+1} = h_{K,i}^{n} \mathbf{u}_{K,i}^{n} - \frac{\Delta t}{m_K} \sum_{e \in \partial K} \left(\mathbf{u}_{K,i}^{n} (\mathcal{F}_{e,i}^{n+1} . \mathbf{n}_{e,K})^{+} - \mathbf{u}_{K_e,i}^{n} (\mathcal{F}_{e,i}^{n+1} . \mathbf{n}_{e,K})^{-} \right) m_e$$
$$- \frac{\Delta t}{m_K} h_{K,i}^{n+1} \sum_{e \in \partial K} \Phi_{e,i}^{n+1} m_e \, ,$$

Flux numériques :

$$\mathcal{F}_{e,i}^{n+1} = h_{e,i}^{n+1} \left(\mathbf{u}_{e,i}^{n} - \gamma \Delta t \frac{\delta \Phi_{e,i}^{n+1}}{m_e} \right) \quad , \quad \text{où} \quad \delta \Phi_{e,i}^{n+1} = \frac{1}{2} \left(\Phi_{K_e,i}^{n+1} - \Phi_{K,i}^{n+1} \right) \mathbf{n}_{e,K} \, .$$

▷ équivalent discret de $h_i \, (\mathbf{u}_i - \delta \mathbf{u}_i)$ avec $\delta \mathbf{u}_i = \gamma \nabla \phi_i$.

Schéma sur la partie potentiel

Schéma sur $\mathcal{E}_K^n$

$$\mathcal{E}_K^{n+1} - \mathcal{E}_K^n + \frac{\Delta t}{m_K} \sum_{e \in \partial K} \sum_{i=1}^{L} \rho_i \big(\mathcal{G}_{\mathcal{E},e,i}^{n+1}.\mathbf{n}_{e,K}\big) m_e - \mathcal{Q}_{\mathcal{E},K} \leq \mathcal{R}_{\mathcal{E},K} + \mathcal{H}_{\mathcal{E},K} \,.$$

$$\mathcal{G}_{\mathcal{E},e,i}^{n+1}.\mathbf{n}_{e,K} = \Phi_{e,i}^{n+1} \mathcal{F}_{e,i}^{n+1}.\mathbf{n}_{e,K} \,,$$

$$\mathcal{H}_{\mathcal{E},K} = \frac{\Delta t}{m_K} \sum_{e \in \partial K} \sum_{i=1}^{L} \rho_i \left(\frac{h_{K_e,i}^{n+1} \mathbf{u}_{K_e,i}^n - h_{K,i}^{n+1} \mathbf{u}_{K,i}^n}{2} \right) .\delta \Phi_{e,i}^{n+1} m_e \,.$$

$$\mathcal{Q}_{\mathcal{E},K} = \frac{\Delta t}{m_K} \sum_{i=1}^{L} \rho_i h_{K,i}^{n+1} \mathbf{u}_{K,i}^n . \sum_{e \in \partial K} \Phi_{e,i}^{n+1} m_e \,,$$

$$\mathcal{R}_{\mathcal{E},K} = \frac{\Delta t}{m_K} \sum_{e \in \partial K} \sum_{i=1}^{L} \rho_i \Big(\mathcal{F}_{e,i}^{n+1} - h_{e,i}^{n+1} \mathbf{u}_{e,i}^n \Big) .\delta \Phi_{e,i}^{n+1} m_e \,.$$

Schéma sur la partie potentiel

Schéma sur $\mathcal{E}_K^n$

$$\mathcal{E}_K^{n+1} - \mathcal{E}_K^n + \frac{\Delta t}{m_K} \sum_{e \in \partial K} \sum_{i=1}^{L} \rho_i \left(\mathcal{G}_{\mathcal{E},e,i}^{n+1} . \mathbf{n}_{e,K} \right) m_e - \mathcal{Q}_{\mathcal{E},K} \leq \mathcal{R}_{\mathcal{E},K} + \mathcal{H}_{\mathcal{E},K} .$$

$$\mathcal{G}_{\mathcal{E},e,i}^{n+1} . \mathbf{n}_{e,K} = \Phi_{e,i}^{n+1} \mathcal{F}_{e,i}^{n+1} . \mathbf{n}_{e,K} ,$$

$$\mathcal{H}_{\mathcal{E},K} = \frac{\Delta t}{m_K} \sum_{e \in \partial K} \sum_{i=1}^{L} \rho_i \left(\frac{h_{K_e,i}^{n+1} \mathbf{u}_{K_e,i}^n - h_{K,i}^{n+1} \mathbf{u}_{K,i}^n}{2} \right) . \delta \Phi_{e,i}^{n+1} m_e ,$$

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$$\mathcal{R}_{\mathcal{E},K} = \frac{\Delta t}{m_K} \sum_{e \in \partial K} \sum_{i=1}^{L} \rho_i \left(\mathcal{F}_{e,i}^{n+1} - h_{e,i}^{n+1} \mathbf{u}_{e,i}^n \right) . \delta \Phi_{e,i}^{n+1} m_e .$$

▷ Version discrète de

$$\partial_t \mathcal{E} + \sum_{i=1}^{L} div \left(\rho_i h_i \phi_i (\mathbf{u}_i - \delta \mathbf{u}_i) \right) - \sum_{i=1}^{L} \rho_i h_i \mathbf{u}_i . \nabla \phi_i = - \sum_{i=1}^{L} \rho_i h_i \delta \mathbf{u}_i . \nabla \phi_i .$$

Schéma sur l'énergie cinétique

Schéma sur $\mathcal{K}_{K,i}^n = \dfrac{1}{2} h_{K,i}^n \|\mathbf{u}_{K,i}^n\|^2$

$$\mathcal{K}_{K,i}^{n+1} - \mathcal{K}_{K,i}^n + \frac{\Delta t}{m_K} \sum_{e\in\partial K} \left(\mathcal{G}_{\mathcal{K},e,i}^{n+1}.\mathbf{n}_{e,K}\right) m_e + \mathcal{Q}_{\mathcal{K},K,i} \leq -\mathcal{H}_{\mathcal{K},K,i} + \mathcal{R}_{\mathcal{K},K,i} \,.$$

$$\mathcal{G}_{\mathcal{K},e,i}^{n+1}.\mathbf{n}_{e,K} = \frac{1}{2}\|\mathbf{u}_{K,i}^n\|^2 \left(\mathcal{F}_{e,i}^{n+1}.\mathbf{n}_{e,K}\right)^+ - \frac{1}{2}\|\mathbf{u}_{K_e,i}^n\|^2 \left(\mathcal{F}_{e,i}^{n+1}.\mathbf{n}_{e,K}\right)^- ,$$

$$\mathcal{H}_{\mathcal{K},K,i} = \frac{\Delta t}{m_K} \sum_{e\in\partial K} \frac{\Delta t}{2} \left(\frac{h_{K_e,i}^{n+1}}{m_{K_e}} - \frac{h_{K,i}^{n+1}}{m_K} \right) \|\delta\Phi_{e,i}^{n+1}\|^2 m_e \,,$$

$$\mathcal{Q}_{\mathcal{K},K,i} = \frac{\Delta t}{m_K} h_{K,i}^{n+1} \mathbf{u}_{K,i}^n . \sum_{e\in\partial K} \Phi_{e,i}^{n+1} m_e \,,$$

$$\mathcal{R}_{\mathcal{K},K,i} = \frac{\Delta t}{m_K} \sum_{e\in\partial K} \Delta t \frac{h_{e,i}^{n+1}}{m_e^{n+1}} \|\delta\Phi_{e,i}^{n+1}\|^2 m_e \,.$$

Schéma sur l'énergie cinétique

Schéma sur $\mathcal{K}^n_{K,i} = \frac{1}{2} h^n_{K,i} \|\mathbf{u}^n_{K,i}\|^2$

$$\mathcal{K}^{n+1}_{K,i} - \mathcal{K}^n_{K,i} + \frac{\Delta t}{m_K} \sum_{e\in\partial K} \left(\mathcal{G}^{n+1}_{\mathcal{K},e,i}.\mathbf{n}_{e,K}\right) m_e + \mathcal{Q}_{\mathcal{K},K,i} \leq -\mathcal{H}_{\mathcal{K},K,i} + \mathcal{R}_{\mathcal{K},K,i}\,.$$

$$\mathcal{G}^{n+1}_{\mathcal{K},e,i}.\mathbf{n}_{e,K} = \frac{1}{2}\|\mathbf{u}^n_{K,i}\|^2 \left(\mathcal{F}^{n+1}_{e,i}.\mathbf{n}_{e,K}\right)^+ - \frac{1}{2}\|\mathbf{u}^n_{K_e,i}\|^2 \left(\mathcal{F}^{n+1}_{e,i}.\mathbf{n}_{e,K}\right)^- ,$$

$$\mathcal{H}_{\mathcal{K},K,i} = \frac{\Delta t}{m_K} \sum_{e\in\partial K} \frac{\Delta t}{2} \left(\frac{h^{n+1}_{K_e,i}}{m_{K_e}} - \frac{h^{n+1}_{K,i}}{m_K} \right) \|\delta\Phi^{n+1}_{e,i}\|^2 m_e\,,$$

$$\mathcal{Q}_{\mathcal{K},K,i} = \frac{\Delta t}{m_K} h^{n+1}_{K,i} \mathbf{u}^n_{K,i}. \sum_{e\in\partial K} \Phi^{n+1}_{e,i} m_e\,,$$

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▷ Version discrète de $\partial_t \mathcal{K}_i + div\left(\mathcal{K}_i(\mathbf{u}_i - \delta\mathbf{u}_i)\right) + h_i \mathbf{u}_i.\nabla\phi_i = 0$.

Bilan

Energie totale

$$
\begin{aligned}
E^{n+1} - E^n &= \sum_K \sum_{i=1}^L m_K \rho_i (\mathcal{K}_{K,i}^{n+1} - \mathcal{K}_{K,i}^n) + \sum_K m_K (\mathcal{E}_K^{n+1} - \mathcal{E}_K^n) \\
&\leq \sum_K \Big(\sum_{i=1}^L m_K \rho_i \mathcal{R}_{\mathcal{K},K,i} + m_K \mathcal{R}_{\mathcal{E},K} \Big) \\
&\leq \Delta t \sum_K \sum_{i=1}^L \sum_{e \in \partial K} \rho_i (1-\gamma) \Delta t \frac{h_{e,i}^{n+1}}{m_e^{n+1}} \|\delta \Phi_{e,i}^{n+1}\|^2 m_e \,.
\end{aligned}
$$

▷ Décroissance de l'énergie numérique $\longrightarrow \gamma \geq 1$.

▷ Résultat identique avec le modèle augmenté (pas d'impact sur la production d'énergie).

Plan

Le schéma

Schéma numérique

$$h_{K,i}^{n+1} = h_{K,i}^{n} - \frac{\Delta t}{m_K} \sum_{e \in \partial K} \mathcal{F}_{e,i}^{n} . \mathbf{n}_{e,K} \, m_e \,,$$

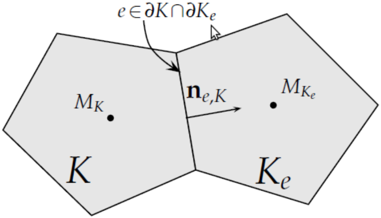
$$h_{K,i}^{n+1} \mathbf{u}_{K,i}^{n+1} = h_{K,i}^{n} \mathbf{u}_{K,i}^{n} - \frac{\Delta t}{m_K} \sum_{e \in \partial K} \Big(\mathbf{u}_{K,i}^{n} \big(\mathcal{F}_{e,i}^{n} . \mathbf{n}_{e,K} \big)^{+} - \mathbf{u}_{K_e,i}^{n} \big(\mathcal{F}_{e,i}^{n} . \mathbf{n}_{e,K} \big)^{-} \Big) m_e$$

$$- \frac{\Delta t}{m_K} h_{K,i}^{n} \sum_{e \in \partial K} \Phi_{e,i}^{n,*} \mathbf{n}_{e,K} \, m_e \,.$$

$$m_K = |K| \quad ; \quad m_e = |e| \,,$$

$$\omega^{\pm} = \frac{1}{2} \left(|\omega| \pm \omega \right) \,,$$

$$\Phi_{e,i}^{n} = \frac{1}{2} \left(\Phi_{K,i}^{n} + \Phi_{K_e,i}^{n} \right) \,.$$



Le schéma

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$$h_{K,i}^{n+1}\mathbf{u}_{K,i}^{n+1} = h_{K,i}^{n}\mathbf{u}_{K,i}^{n} - \frac{\Delta t}{m_K} \sum_{e \in \partial K} \left(\mathbf{u}_{K,i}^{n} \left(\mathcal{F}_{e,i}^{n} . \mathbf{n}_{e,K}\right)^{+} - \mathbf{u}_{K_e,i}^{n} \left(\mathcal{F}_{e,i}^{n} . \mathbf{n}_{e,K}\right)^{-} \right) m_e$$

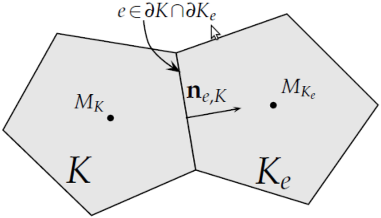
$$- \frac{\Delta t}{m_K} h_{K,i}^{n} \sum_{e \in \partial K} \Phi_{e,i}^{n,*} \mathbf{n}_{e,K}\, m_e \,.$$

$$m_K = |K| \quad ; \quad m_e = |e| \,,$$

$$\omega^{\pm} = \frac{1}{2} \left(|\omega| \pm \omega \right) \,,$$

$$\Phi_{e,i}^{n} = \frac{1}{2} \left(\Phi_{K,i}^{n} + \Phi_{K_e,i}^{n} \right) \,.$$

- $\mathcal{F}_{e,i}^{n} = (h\mathbf{u})_{e,i}^{n} - \Pi_{e,i}^{n} \quad .$



Le schéma

Schéma numérique

$$h_{K,i}^{n+1} = h_{K,i}^{n} - \frac{\Delta t}{m_K} \sum_{e \in \partial K} \mathcal{F}_{e,i}^{n} . \mathbf{n}_{e,K}\, m_e \,,$$

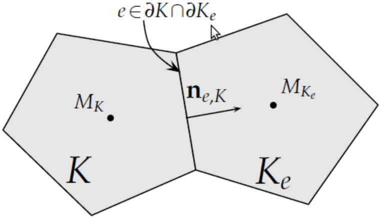
$$h_{K,i}^{n+1}\mathbf{u}_{K,i}^{n+1} = h_{K,i}^{n}\mathbf{u}_{K,i}^{n} - \frac{\Delta t}{m_K} \sum_{e \in \partial K} \left(\mathbf{u}_{K,i}^{n} \left(\mathcal{F}_{e,i}^{n} . \mathbf{n}_{e,K}\right)^{+} - \mathbf{u}_{K_e,i}^{n} \left(\mathcal{F}_{e,i}^{n} . \mathbf{n}_{e,K}\right)^{-} \right) m_e$$

$$- \frac{\Delta t}{m_K} h_{K,i}^{n} \sum_{e \in \partial K} \Phi_{e,i}^{n,*} \mathbf{n}_{e,K}\, m_e \,.$$

$$m_K = |K| \quad ; \quad m_e = |e| \,,$$

$$\omega^{\pm} = \frac{1}{2}\left(|\omega| \pm \omega\right) \,,$$

$$\Phi_{e,i}^{n} = \frac{1}{2}\left(\Phi_{K,i}^{n} + \Phi_{K_e,i}^{n}\right) \,.$$



- $\mathcal{F}_{e,i}^{n} = (h\mathbf{u})_{e,i}^{n} - \Pi_{e,i}^{n}$.
- $\Phi_{e,i}^{n,*} = \Phi_{e,i}^{n} - \Lambda_{e,i}^{n}$.

Termes régularisants

Flux régularisés

$$\mathcal{F}_{e,i}^n = (h\mathbf{u})_{e,i}^n - \Pi_{e,i}^n = (h\mathbf{u})_{e,i}^n - \gamma \Delta t (h\mu)_{e,i}^{n+1/2} \delta\Phi_{e,i}^n \quad \rightarrow \text{correction en } {\color{red}\nabla\phi}$$

$$\text{avec } (h\mu)_{e,i}^{n+1/2} = \frac{1}{2}\left(\frac{(h_{K,i}^n)^2}{h_{K,i}^{n+1}} \frac{m_{\partial K}}{m_K} + \frac{(h_{K_e,i}^n)^2}{h_{K_e,i}^{n+1}} \frac{m_{\partial K_e}}{m_{K_e}} \right).$$

Termes régularisants

Flux régularisés

$$\mathcal{F}^n_{e,i} = (h\mathbf{u})^n_{e,i} - \Pi^n_{e,i} = (h\mathbf{u})^n_{e,i} - \gamma \Delta t (h\mu)^{n+1/2}_{e,i} \delta \Phi^n_{e,i} \quad \rightarrow \text{correction en } \nabla \phi$$

$$\text{avec } (h\mu)^{n+1/2}_{e,i} = \frac{1}{2} \left(\frac{(h^n_{K,i})^2}{h^{n+1}_{K,i}} \frac{m_{\partial K}}{m_K} + \frac{(h^n_{K_e,i})^2}{h^{n+1}_{K_e,i}} \frac{m_{\partial K_e}}{m_{K_e}} \right).$$

Potentiel régularisé

$$\Phi^{n,*}_{e,i} = \Phi^n_{e,i} - \Lambda^n_{e,i} = \Phi^n_{e,i} - \alpha \nu \Delta t \mu_e \delta (h\mathbf{u})^n_{e,i} \quad \rightarrow \text{correction en } \nabla . h u$$

$$\text{avec } \mu_e = \frac{1}{2} \left(\frac{m_{\partial K}}{m_K} + \frac{m_{\partial K_e}}{m_{K_e}} \right).$$

Termes régularisants

Flux régularisés

$$\mathcal{F}^n_{e,i} = (h\mathbf{u})^n_{e,i} - \Pi^n_{e,i} = (h\mathbf{u})^n_{e,i} - \gamma \Delta t (h\mu)^{n+1/2}_{e,i} \delta \Phi^n_{e,i} \quad \rightarrow \text{correction en } \nabla\phi$$

$$\text{avec } (h\mu)^{n+1/2}_{e,i} = \frac{1}{2}\left(\frac{(h^n_{K,i})^2}{h^{n+1}_{K,i}} \frac{m_{\partial K}}{m_K} + \frac{(h^n_{K_e,i})^2}{h^{n+1}_{K_e,i}} \frac{m_{\partial K_e}}{m_{K_e}} \right).$$

Potentiel régularisé

$$\Phi^{n,*}_{e,i} = \Phi^n_{e,i} - \Lambda^n_{e,i} = \Phi^n_{e,i} - \alpha \nu \Delta t \mu_e \delta (h\mathbf{u})^n_{e,i} \quad \rightarrow \text{correction en } \nabla . h\mathbf{u}$$

$$\text{avec } \mu_e = \frac{1}{2}\left(\frac{m_{\partial K}}{m_K} + \frac{m_{\partial K_e}}{m_{K_e}} \right).$$

Conditions sur γ, α

$$\gamma, \alpha \in \mathcal{S} := \left\{ x\,,\, p(x) = (k_e)^2 x^2 - x + 1 \leq 0 \right\}$$

Cas 1d barotrope

Conditions

$$\gamma, \alpha \in \mathcal{S} := \left\{ x \, , p(x) = (k_e)^2 x^2 - x + 1 \leq 0 \, \right\} \quad , \quad k_e = 2 \frac{\Delta t}{\Delta x} c_e \, .$$

$$\Delta = 1 - 4k_e^2 \geq 0 \Rightarrow \frac{\Delta t}{\Delta x} c_e \leq \frac{1}{4} \text{ (Condition CFL)} \, ,$$

$$x^{\pm} = \frac{1 \pm \sqrt{1 - 4k_e^2}}{2k_e^2} ,$$

$$\mathcal{S} = \left[x^-, x^+ \right] , \text{ condition à } \textbf{priori locale}.$$

Cas 1d barotrope

Conditions

$$\gamma, \alpha \in \mathcal{S} := \left\{ x \,, p(x) = (k_e)^2 x^2 - x + 1 \leq 0 \right\} \quad , \quad k_e = 2\frac{\Delta t}{\Delta x} c_e \,.$$

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$$x^{\pm} = \frac{1 \pm \sqrt{1 - 4k_e^2}}{2k_e^2},$$

$$\mathcal{S} = \left[x^-, x^+ \right] \text{, condition à } \textbf{priori locale}.$$

Choix des paramètres de diffusion

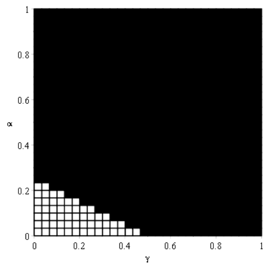
▷ $2 \in \mathcal{S} : \alpha = \gamma = 2$ permet d'assurer la stabilité du schéma.
▷ Expérimentation numérique : schémas trop diffusifs en pratique.
→ On cherche des conditions relaxées en régime bas-Froude :

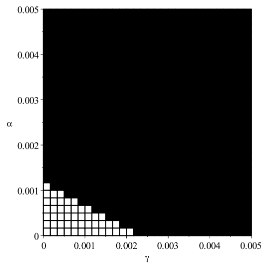
$$\alpha = \gamma = 1/2$$

→ Analyse de stabilité linéaire : importance capitale du schéma temporel.

Cartographies de stabilité

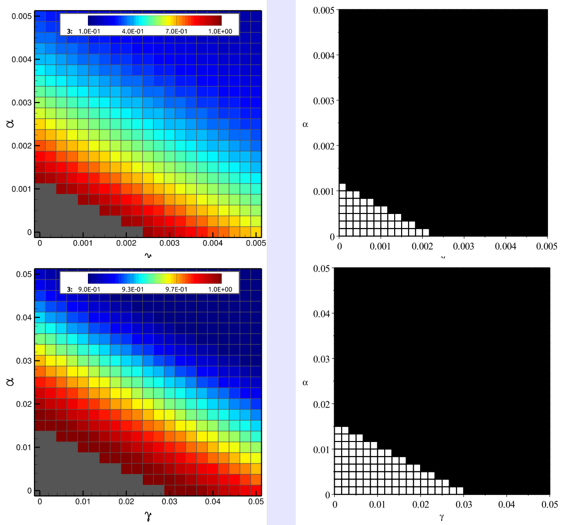
Schémas d'ordre 1 et RK2 en temps





Cartographies de stabilité

Numérique Vs Analyse linéaire. Ordre 1 et MUSCL en espace



Analyse asymptotique

Interprétation semi discrète

$$\left\{\begin{array}{rl} h_i^{n+1} - h_i^n = & -\Delta t \partial_x \left(hu_i\right)^n + (\Delta t)^2 \gamma \partial_x \left(h_i \dfrac{\partial_x \phi_i}{\epsilon^2}\right)^n \\ hu_i^{n+1} - hu_i^n = & -\Delta t \left(\partial_x \left(\bar{u}_i hu_i^*\right)\right)^n \\ & -\Delta t \left(h_i \dfrac{\partial_x \phi_i}{\epsilon^2}\right)^n + (\Delta t)^2 \alpha \left(h_i \dfrac{\partial_{xx} \left(hu_i\right)}{\epsilon^2}\right)^n \end{array}\right.$$

avec $hu_i^* = hu_i - \Delta t \gamma \left(h_i \dfrac{\partial_x \phi_i}{\epsilon^2}\right)$ et $\bar{u}_i = u_i + \mathcal{O}(\Delta x)$.

Analyse temps court $t = \epsilon \tau$

Equation de la masse

$$\begin{array}{ccccccc} h_i^{n+1} & - & h_i^n & = & -\epsilon \Delta \tau \partial_x \left(hu_i\right)^n & + & (\Delta \tau)^2 \gamma \partial_x \left(h_i \partial_x \phi_i\right)^n , \\ h_i^n & - & h_i^{n-1} & = & -\epsilon \Delta \tau \partial_x \left(hu_i\right)^{n-1} & + & (\Delta \tau)^2 \gamma \partial_x \left(h_i \partial_x \phi_i\right)^{n-1} , \end{array}$$

Analyse asymptotique

Interprétation semi discrète

$$\left\{\begin{array}{ll} h_i^{n+1} - h_i^n = & -\Delta t \partial_x \left(hu_i\right)^n + (\Delta t)^2 \gamma \partial_x \left(h_i \dfrac{\partial_x \phi_i}{\epsilon^2}\right)^n \\ hu_i^{n+1} - hu_i^n = & -\Delta t \left(\partial_x \left(\bar{u}_i hu_i^*\right)\right)^n \\ & -\Delta t \left(h_i \dfrac{\partial_x \phi_i}{\epsilon^2}\right)^n + (\Delta t)^2 \alpha \left(h_i \dfrac{\partial_{xx} \left(hu_i\right)}{\epsilon^2}\right)^n \end{array}\right.$$

avec $hu_i^* = hu_i - \Delta t \gamma \left(h_i \dfrac{\partial_x \phi_i}{\epsilon^2}\right)$ et $\bar{u}_i = u_i + \mathcal{O}(\Delta x)$.

Analyse temps court $t = \epsilon\tau$

Equation de la masse

$$\begin{array}{ccccccc} h_i^{n+1} & - & h_i^n & = & -\epsilon \Delta \tau \partial_x \left(hu_i\right)^n & + & (\Delta \tau)^2 \gamma \partial_x \left(h_i \partial_x \phi_i\right)^n, \\ h_i^n & - & h_i^{n-1} & = & -\epsilon \Delta \tau \partial_x \left(hu_i\right)^{n-1} & + & (\Delta \tau)^2 \gamma \partial_x \left(h_i \partial_x \phi_i\right)^{n-1}, \end{array}$$

$$\frac{h_i^{n+1} - 2h_i^n + h_i^{n-1}}{(\Delta \tau)^2} = -\frac{\epsilon}{\Delta \tau}\left[\partial_x \left(hu_i^n - hu_i^{n-1}\right)\right] + \gamma \left[\partial_x \left(\left(h_i \partial_x \phi_i\right)^n - \left(h_i \partial_x \phi_i\right)^{n-1}\right)\right].$$

Analyse asymptotique

Interprétation semi discrète

$$\left\{\begin{array}{ll} h_i^{n+1} - h_i^n = & -\Delta t \partial_x \left(hu_i\right)^n + (\Delta t)^2 \gamma \partial_x \left(h_i \dfrac{\partial_x \phi_i}{\epsilon^2}\right)^n \\ hu_i^{n+1} - hu_i^n = & -\Delta t \left(\partial_x \left(\bar{u}_i hu_i^*\right)\right)^n \\ & -\Delta t \left(h_i \dfrac{\partial_x \phi_i}{\epsilon^2}\right)^n + (\Delta t)^2 \alpha \left(h_i \dfrac{\partial_{xx} \left(hu_i\right)}{\epsilon^2}\right)^n \end{array}\right.$$

avec $hu_i^* = hu_i - \Delta t \gamma \left(h_i \dfrac{\partial_x \phi_i}{\epsilon^2}\right)$ et $\bar{u}_i = u_i + \mathcal{O}(\Delta x)$.

Analyse temps court $t = \epsilon \tau$

Equation de la masse

$$\begin{array}{ccccccc} h_i^{n+1} & - & h_i^n & = & -\epsilon \Delta \tau \partial_x \left(hu_i\right)^n & + & (\Delta \tau)^2 \gamma \partial_x \left(h_i \partial_x \phi_i\right)^n , \\ h_i^n & - & h_i^{n-1} & = & -\epsilon \Delta \tau \partial_x \left(hu_i\right)^{n-1} & + & (\Delta \tau)^2 \gamma \partial_x \left(h_i \partial_x \phi_i\right)^{n-1} , \end{array}$$

$$\frac{h_i^{n+1} - 2h_i^n + h_i^{n-1}}{(\Delta \tau)^2} = -\frac{\epsilon}{\Delta \tau} \left[\partial_x \left(hu_i^n - hu_i^{n-1}\right)\right] + \underbrace{\gamma \left[\partial_x \left(\left(h_i \partial_x \phi_i\right)^n - \left(h_i \partial_x \phi_i\right)^{n-1}\right)\right]}_{\mathcal{O}(\Delta \tau)} .$$

Analyse asymptotique

Interprétation semi discrète

$$\begin{cases} h_i^{n+1} - h_i^n = & -\Delta t \partial_x (hu_i)^n + (\Delta t)^2 \gamma \partial_x \left(h_i \dfrac{\partial_x \phi_i}{\epsilon^2} \right)^n \\ hu_i^{n+1} - hu_i^n = & -\Delta t \left(\partial_x \left(\bar{u}_i hu_i^* \right) \right)^n \\ & -\Delta t \left(h_i \dfrac{\partial_x \phi_i}{\epsilon^2} \right)^n + (\Delta t)^2 \alpha \left(h_i \dfrac{\partial_{xx} (hu_i)}{\epsilon^2} \right)^n \end{cases}$$

avec $hu_i^* = hu_i - \Delta t \gamma \left(h_i \dfrac{\partial_x \phi_i}{\epsilon^2} \right)$ et $\bar{u}_i = u_i + \mathcal{O}(\Delta x)$.

Analyse temps court $t = \epsilon \tau$

$$\frac{h_i^{n+1} - 2h_i^n + h_i^{n-1}}{(\Delta \tau)^2} = -\frac{\epsilon}{\Delta \tau} \left[\partial_x \left(hu_i^n - hu_i^{n-1} \right) \right] + \mathcal{O}(\Delta \tau).$$

$$\frac{\epsilon}{\Delta \tau} \left(hu_i^n - hu_i^{n-1} \right) = -\epsilon^2 \left(\partial_x \left(\bar{u}_i hu_i^* \right) \right)^{n-1}$$

$$-\epsilon^2 \left(h_i \frac{\partial_x \phi_i}{\epsilon^2} \right)^{n-1} + \epsilon \Delta \tau \alpha \left(h_i \partial_{xx} (hu_i) \right)^{n-1}$$

Analyse asymptotique

Interprétation semi discrète

$$\left\{\begin{array}{ll} h_i^{n+1} - h_i^n = & -\Delta t \partial_x \left(hu_i\right)^n + (\Delta t)^2 \gamma \partial_x \left(h_i \dfrac{\partial_x \phi_i}{\epsilon^2}\right)^n \\ hu_i^{n+1} - hu_i^n = & -\Delta t \left(\partial_x \left(\bar{u}_i hu_i^*\right)\right)^n \\ & -\Delta t \left(h_i \dfrac{\partial_x \phi_i}{\epsilon^2}\right)^n + (\Delta t)^2 \alpha \left(h_i \dfrac{\partial_{xx} \left(hu_i\right)}{\epsilon^2}\right)^n \end{array}\right.$$

avec $hu_i^* = hu_i - \Delta t \gamma \left(h_i \dfrac{\partial_x \phi_i}{\epsilon^2}\right)$ et $\bar{u}_i = u_i + \mathcal{O}(\Delta x)$.

Analyse temps court $t = \epsilon\tau$

$$\frac{h_i^{n+1} - 2h_i^n + h_i^{n-1}}{(\Delta\tau)^2} = -\frac{\epsilon}{\Delta\tau}\left[\partial_x \left(hu_i^n - hu_i^{n-1}\right)\right] + \mathcal{O}(\Delta\tau).$$

$$\frac{\epsilon}{\Delta\tau}\left(hu_i^n - hu_i^{n-1}\right) = -\epsilon^2 \left(\partial_x \left(\bar{u}_i hu_i^*\right)\right)^{n-1} - \left(h_i \partial_x \phi_i\right)^n + \mathcal{O}(\Delta\tau).$$

Analyse asymptotique

Interprétation semi discrète

$$\left\{\begin{array}{rl} h_i^{n+1} - h_i^n = & -\Delta t \partial_x \left(hu_i\right)^n + (\Delta t)^2 \gamma \partial_x \left(h_i \dfrac{\partial_x \phi_i}{\epsilon^2}\right)^n \\ hu_i^{n+1} - hu_i^n = & -\Delta t \left(\partial_x \left(\bar{u}_i hu_i^*\right)\right)^n \\ & -\Delta t \left(h_i \dfrac{\partial_x \phi_i}{\epsilon^2}\right)^n + (\Delta t)^2 \alpha \left(h_i \dfrac{\partial_{xx}\left(hu_i\right)}{\epsilon^2}\right)^n \end{array}\right.$$

avec $hu_i^* = hu_i - \Delta t \gamma \left(h_i \dfrac{\partial_x \phi_i}{\epsilon^2}\right)$ et $\bar{u}_i = u_i + \mathcal{O}(\Delta x)$.

Analyse temps court $t = \epsilon\tau$

$$\frac{h_i^{n+1} - 2h_i^n + h_i^{n-1}}{(\Delta\tau)^2} = -\frac{\epsilon}{\Delta\tau}\left[\partial_x\left(hu_i^n - hu_i^{n-1}\right)\right] + \mathcal{O}(\Delta\tau).$$

$$\frac{\epsilon}{\Delta\tau}\left(hu_i^n - hu_i^{n-1}\right) = -\epsilon^2\left(\partial_x\left(\bar{u}_i hu_i^*\right)\right)^{n-1} - \left(h_i \partial_x \phi_i\right)^n + \mathcal{O}(\Delta\tau).$$

$$\left.\begin{array}{l} \epsilon^2 hu_i^* = \underset{\epsilon\to 0}{\mathcal{O}}(\epsilon^2) + \mathcal{O}(\Delta\tau) \\ \bar{u}_i = \mathcal{O}(1) \end{array}\right\} \Rightarrow \epsilon^2\left(\partial_x\left(\bar{u}_i hu_i^*\right)\right)^{n-1} = \underset{\epsilon\to 0}{\mathcal{O}}(\epsilon^2) + \mathcal{O}(\Delta\tau).$$

Analyse asymptotique

Interprétation semi discrète

$$\begin{cases} h_i^{n+1} - h_i^n = & -\Delta t \partial_x \left(hu_i\right)^n + (\Delta t)^2 \gamma \partial_x \left(h_i \dfrac{\partial_x \phi_i}{\epsilon^2}\right)^n \\ hu_i^{n+1} - hu_i^n = & -\Delta t \left(\partial_x \left(\bar{u}_i hu_i^*\right)\right)^n \\ & -\Delta t \left(h_i \dfrac{\partial_x \phi_i}{\epsilon^2}\right)^n + (\Delta t)^2 \alpha \left(h_i \dfrac{\partial_{xx}\left(hu_i\right)}{\epsilon^2}\right)^n \end{cases}$$

avec $hu_i^* = hu_i - \Delta t \gamma \left(h_i \dfrac{\partial_x \phi_i}{\epsilon^2}\right)$ et $\bar{u}_i = u_i + \mathcal{O}(\Delta x)$.

Analyse temps court $t = \epsilon \tau$

$$\frac{h_i^{n+1} - 2h_i^n + h_i^{n-1}}{(\Delta \tau)^2} = \partial_x \left(h_i \partial_x \phi_i\right)^n + \underset{\epsilon \to 0}{\mathcal{O}}\left(\epsilon^2\right) + \mathcal{O}(\Delta \tau) .$$

Plan

Ondes de gravité 2d en régime linéaire

Configuration

- 5 couches .
- $\rho_i = 1000 + 10(i-1)$.
- $\ell = 100km$, $h0 = 5000m$.

Condition initiale

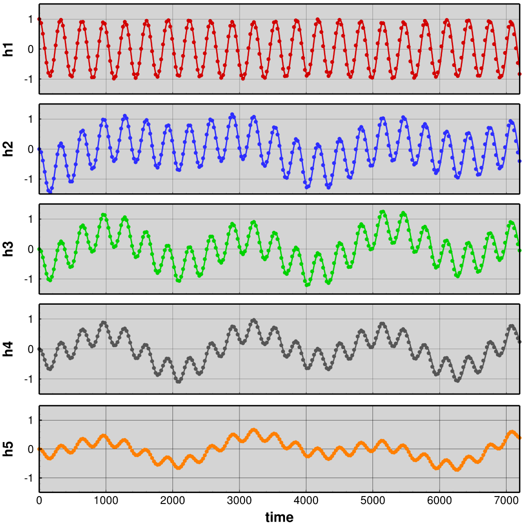
Perturbation de surface :

- $\delta h \left(cos(kx) + cos(ky) \right)$,

  $\delta h = 1m$, $k = \dfrac{2\pi}{\ell}$.

Paramètres numériques

- grille 41×41.
- $\alpha = \beta = 0.1$.
- Ordre 2 temps (RK2) et espace (MUSCL) .



Transport de vortex sur β plan : calibration

Paramètres

$$L = 2 \qquad R_{earth} = 6367442,76m$$
$$h_0 = 5000m \qquad \Omega = 2\pi/(3600/24)$$
$$h_1 = 2500m \qquad f_0 = 2\Omega sin(\theta) \simeq 9,25410^{-5}$$
$$\theta = 38.5^\circ \qquad \beta = 2\Omega cos(\theta)/R_{earth} \simeq 1,78810^{-11}$$

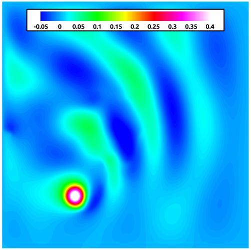
- Domaine $[-1000, 1000] \times [-1000, 1000]$.
- Condition initiale déterminée à partir de la distribution de densité.

Coriolis : splitting

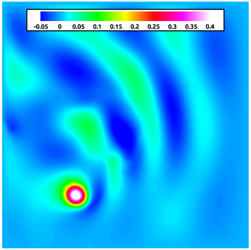
Approximation β plan : $f = f_0 + \beta y$.

$$u^{n+1} = cos(f\Delta t)u^n + sin(f\Delta t)v^n\,,$$
$$v^{n+1} = cos(f\Delta t)v^n - sin(f\Delta t)u^n\,.$$

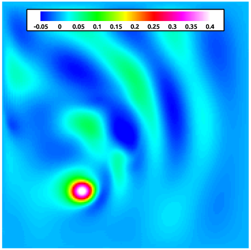
Transport de vortex sur β plan : déviation sur 100 jours



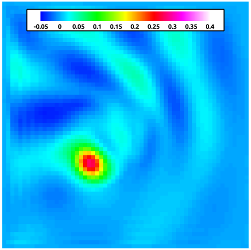
(a) $\Delta x = 2km$



(b) $\Delta x = 5km$

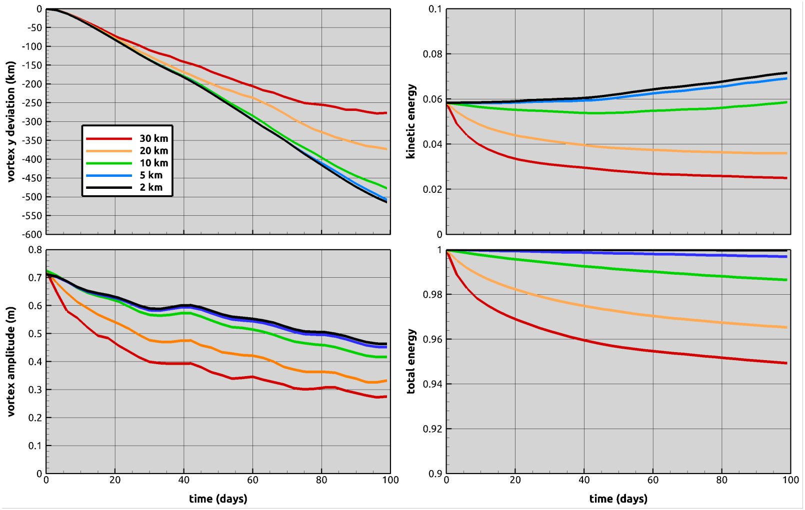


(c) $\Delta x = 10km$



(d) $\Delta x = 20km$

Transport de vortex sur β plan : déviation sur 100 jours



Plan

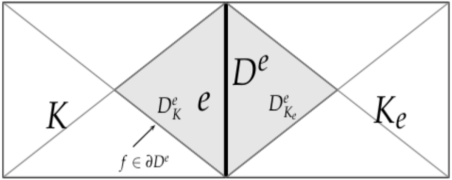
Extension aux mailles décalées

Formalisme : **R. Herbin, W. Kheriji, J.C. Latché, K.Saleh** (2013, 2014).

Schéma

$$h_{K,i}^{n+1} = h_{K,i}^{n} - \frac{\Delta t}{m_K} \sum_{e \in \partial K} \mathcal{F}_{e,i}^{n+1} . \mathbf{n}_{e,K} m_e$$

$$h_{D^e,i}^{n+1} \mathbf{u}_{D^e,i}^{n+1} = h_{D^e,i}^{n} \mathbf{u}_{D^e,i}^{n} - \frac{\Delta t}{m_{D^e}} \sum_{f \in \partial D^e} \Big(\mathbf{u}_{D^e,i}^{n} (\mathcal{F}_{f,i}^{n+1} . \mathbf{n}_{f,D^e})^{+} - \mathbf{u}_{D_f^e,i}^{n} (\mathcal{F}_{f,i}^{n+1} . \mathbf{n}_{f,D^e})^{-} \Big) m_f$$
$$- \frac{\Delta t}{m_{D^e}} h_{D^e,i}^{n+1} \nabla \Phi_{e,i} .$$



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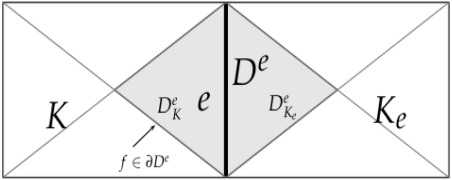
$$h_{K,i}^{n+1} = h_{K,i}^{n} - \frac{\Delta t}{m_K} \sum_{e \in \partial K} \mathcal{F}_{e,i}^{n+1} . \mathbf{n}_{e,K} m_e$$

$$h_{D^e,i}^{n+1} \mathbf{u}_{D^e,i}^{n+1} = h_{D^e,i}^{n} \mathbf{u}_{D^e,i}^{n} - \frac{\Delta t}{m_{D^e}} \sum_{f \in \partial D^e} \left(\mathbf{u}_{D^e,i}^{n} \left(\mathcal{F}_{f,i}^{n+1} . \mathbf{n}_{f,D^e} \right)^{+} - \mathbf{u}_{D^e_f,i}^{n} \left(\mathcal{F}_{f,i}^{n+1} . \mathbf{n}_{f,D^e} \right)^{-} \right) m_f$$
$$- \frac{\Delta t}{m_{D^e}} h_{D^e,i}^{n+1} \nabla \Phi_{e,i} \, .$$

▷ **Flux numériques :** $\mathcal{F}_{e,i}^{n+1} = h_{D^e,i}^{n+1} (\mathbf{u}_{D^e,i}^{n} - \delta \mathbf{u}_{D^e,i}^{n})$.

▷ **Gradient de pression :** $\Phi_{e,i}$ à déterminer.



Extension aux mailles décalées

Formalisme : **R. Herbin, W. Kheriji, J.C. Latché, K.Saleh** (2013, 2014).

Schéma

$$h^{n+1}_{K,i} = h^n_{K,i} - \frac{\Delta t}{m_K} \sum_{e \in \partial K} \mathcal{F}^{n+1}_{e,i} . \mathbf{n}_{e,K} m_e$$

$$h^{n+1}_{D^e,i} \mathbf{u}^{n+1}_{D^e,i} = h^n_{D^e,i} \mathbf{u}^n_{D^e,i} - \frac{\Delta t}{m_{D^e}} \sum_{f \in \partial D^e} \left(\mathbf{u}^n_{D^e,i} \left(\mathcal{F}^{n+1}_{f,i} . \mathbf{n}_{f,D^e} \right)^+ - \mathbf{u}^n_{D^e_f,i} \left(\mathcal{F}^{n+1}_{f,i} . \mathbf{n}_{f,D^e} \right)^- \right) m_f$$

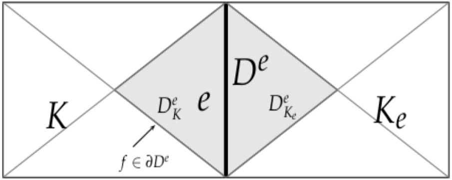
$$- \frac{\Delta t}{m_{D^e}} h^{n+1}_{D^e,i} \nabla \Phi_{e,i} .$$

- ▷ **Flux numériques :** $\mathcal{F}^{n+1}_{e,i} = h^{n+1}_{D^e,i} (\mathbf{u}^n_{D^e,i} - \delta \mathbf{u}^n_{D^e,i})$.
- ▷ **Gradient de pression :** $\Phi_{e,i}$ à déterminer.

Hauteur d'eau auxiliaire

$$|D^e| h^{n+1}_{D^e,i} = |D^e_K| h^{n+1}_{K,i} + |D^e_{K_e}| h^{n+1}_{K_e,i} ,$$

$$h^{n+1}_{D^e,i} - h^n_{D^e,i} = - \frac{\Delta t}{m_{D^e}} \sum_{f \in \partial D^e} \mathcal{F}^{n+1}_{f,i} . \mathbf{n}_{f,D^e} m_f .$$



Schémas d'énergie : partie cinétique

Schéma sur $\mathcal{K}^n_{D^e,i} = \frac{1}{2} h^n_{D^e,i} \|\mathbf{u}^n_{D^e,i}\|^2$

$$\mathcal{K}^{n+1}_{D^e,i} - \mathcal{K}^n_{D^e,i} \leq -\frac{\Delta t}{m_{D^e}} \sum_{f \in \partial D^e} \left(\mathcal{G}^{n+1}_{\mathcal{K},f,i} . \mathbf{n}_{f,D^e} \right) m_f - \mathcal{Q}_{\mathcal{K},e,i} + \mathcal{R}_{\mathcal{K},e,i} \,.$$

$$\mathcal{G}^{n+1}_{\mathcal{K},f,i} . \mathbf{n}_{f,D^e} = \frac{1}{2} \|\mathbf{u}^n_{D^e,i}\|^2 \left(\mathcal{F}^{n+1}_{f,i} . \mathbf{n}_{f,D^e} \right)^+ - \frac{1}{2} \|\mathbf{u}^n_{D^e_f,i}\|^2 \left(\mathcal{F}^{n+1}_{f,i} . \mathbf{n}_{f,D^e} \right)^- ,$$

$$\mathcal{Q}_{\mathcal{K},e,i} = \Delta t \, h^{n+1}_{D^e,i} \mathbf{u}^n_{D^e,i} . (\nabla \Phi)_{e,i} \,,$$

$$\mathcal{R}_{\mathcal{K},e,i} = h^{n+1}_{D^e,i} (\Delta t)^2 \|(\nabla \Phi)_{e,i}\|^2 \,.$$

Schémas d'énergie : partie potentiel

Schéma sur $\mathcal{E}$

$$\mathcal{E}_K^{n+1} - \mathcal{E}_K^n \leq -\frac{\Delta t}{m_K} \sum_{e\in\partial K} \sum_{i=1}^{L} \left(\mathcal{G}^n_{\mathcal{E},e,i} . \mathbf{n}_{e,K} \right) m_e + \mathcal{Q}_{\mathcal{E},K} - \mathcal{R}_{\mathcal{E},K} \,.$$

$$\mathcal{G}^n_{\mathcal{E},e,i} . \mathbf{n}_{e,K} = \Phi^{n+1}_{e,i} \mathcal{F}^{n+1}_{e,i} . \mathbf{n}_{e,K} \,,$$

$$\mathcal{Q}_{\mathcal{E},K} = \frac{\Delta t}{m_K} \sum_{e\in\partial K} \sum_{i=1}^{L} h^{n+1}_{D^e,i} \mathbf{u}^n_{e,i} . \delta\Phi^{n+1}_{e,i} m_e \,,$$

$$\mathcal{R}_{\mathcal{E},K} = \frac{\Delta t}{m_K} \sum_{e\in\partial K} \sum_{i=1}^{L} h^{n+1}_{D^e,i} \delta\mathbf{u}^n_{e,i} . \delta\Phi^{n+1}_{e,i} m_e \,.$$

Bilan

Energie totale

$$E^{n+1} - E^n \leq \sum_{D^e} \sum_{i=1}^{L} \rho_i m_{D^e} (\mathcal{K}_{D^e,i}^{n+1} - \mathcal{K}_{D^e,i}^{n}) + \sum_K m_K (\mathcal{E}_K^{n+1} - \mathcal{E}_K^{n}) \,.$$

Bilan

Energie totale

$$E^{n+1} - E^n \leq \sum_{D^e} \sum_{i=1}^{L} \rho_i m_{D^e} \left(-\mathcal{Q}_{\mathcal{K},e,i} + \mathcal{R}_{\mathcal{K},e,i} \right) + \sum_{K} m_K \left(\mathcal{Q}_{\mathcal{E},K} - \mathcal{R}_{\mathcal{E},K} \right) .$$

Bilan

Energie totale

$$E^{n+1} - E^n \leq \sum_{D^e} \sum_{i=1}^{L} \rho_i m_{D^e} \left(-\mathcal{Q}_{\mathcal{K},e,i} + \mathcal{R}_{\mathcal{K},e,i} \right) + \sum_K m_K \left(\mathcal{Q}_{\mathcal{E},K} - \mathcal{R}_{\mathcal{E},K} \right).$$

▷ Termes en $\mathcal{Q}$:

$$\sum_{D^e} \sum_{i=1}^{L} \rho_i m_{D^e} \mathcal{Q}_{\mathcal{K},e,i} = \Delta t \sum_{D^e} \sum_{i=1}^{L} \rho_i m_{D^e} h_{D^e,i}^{n+1} (\nabla \Phi)_{e,i} . \mathbf{u}_{D^e,i}^n$$

$$\sum_K m_K \mathcal{Q}_{\mathcal{E},K} = \Delta t \sum_K \sum_{e \in \partial K} \sum_{i=1}^{L} \rho_i h_{D^e,i}^{n+1} \mathbf{u}_{e,i}^n . \delta \Phi_{e,i}^{n+1} m_e$$

Bilan

Energie totale

$$E^{n+1} - E^n \leq \sum_{D^e} \sum_{i=1}^{L} \rho_i m_{D^e} \left(-\mathcal{Q}_{\mathcal{K},e,i} + \mathcal{R}_{\mathcal{K},e,i} \right) + \sum_{K} m_K \left(\mathcal{Q}_{\mathcal{E},K} - \mathcal{R}_{\mathcal{E},K} \right).$$

▷ Termes en $\mathcal{Q}$:

$$\sum_{D^e} \sum_{i=1}^{L} \rho_i m_{D^e} \mathcal{Q}_{\mathcal{K},e,i} = \Delta t \sum_{D^e} \sum_{i=1}^{L} \rho_i m_{D^e} h^{n+1}_{D^e,i} (\nabla \Phi)_{e,i} . \mathbf{u}^n_{D^e,i}$$

$$\sum_{K} m_K \mathcal{Q}_{\mathcal{E},K} = \Delta t \sum_{K} \sum_{e \in \partial K} \sum_{i=1}^{L} \rho_i h^{n+1}_{D^e,i} \mathbf{u}^n_{e,i} . \delta \Phi^{n+1}_{e,i} m_e$$

Compensation des termes : $(\nabla \Phi)_{e,i} = \dfrac{m_e}{m_{D^e}} \left(\Phi^{n+1}_{K_e,i} - \Phi^{n+1}_{K,i} \right) \mathbf{n}_{e,K}.$

Bilan

Energie totale

$$E^{n+1} - E^n \le \sum_{D^e} \sum_{i=1}^{L} \rho_i m_{D^e} \left(-\mathcal{Q}_{\mathcal{K},e,i} + \mathcal{R}_{\mathcal{K},e,i} \right) + \sum_K m_K \left(\mathcal{Q}_{\mathcal{E},K} - \mathcal{R}_{\mathcal{E},K} \right) .$$

▷ Termes en $\mathcal{R}$:

$$\sum_{D^e} \sum_{i=1}^{L} m_{D^e} \mathcal{R}_{\mathcal{K},e,i} = \sum_{D^e} \sum_{i=1}^{L} m_{D^e} h_{D^e,i}^{n+1} (\Delta t)^2 \|(\nabla \Phi)_{e,i}\|^2$$

$$\sum_K m_K \mathcal{R}_{\mathcal{E},K} = \Delta t \sum_K \sum_{e \in \partial K} \sum_{i=1}^{L} h_{D^e,i}^{n+1} \delta \mathbf{u}_{e,i}^n . \delta \Phi_{e,i}^{n+1} m_e$$

Bilan

Energie totale

$$E^{n+1} - E^n \leq \sum_{D^e} \sum_{i=1}^{L} \rho_i m_{D^e} \left(-\mathcal{Q}_{\mathcal{K},e,i} + \mathcal{R}_{\mathcal{K},e,i} \right) + \sum_K m_K \left(\mathcal{Q}_{\mathcal{E},K} - \mathcal{R}_{\mathcal{E},K} \right).$$

▷ Termes en $\mathcal{R}$:

$$\sum_{D^e} \sum_{i=1}^{L} m_{D^e} \mathcal{R}_{\mathcal{K},e,i} = \sum_{D^e} \sum_{i=1}^{L} m_{D^e} h_{D^e,i}^{n+1} (\Delta t)^2 \| (\nabla \Phi)_{e,i} \|^2$$

$$\sum_K m_K \mathcal{R}_{\mathcal{E},K} = \Delta t \sum_K \sum_{e \in \partial K} \sum_{i=1}^{L} h_{D^e,i}^{n+1} \delta \mathbf{u}_{e,i}^n . \delta \Phi_{e,i}^{n+1} m_e$$

Restes négatifs : $\delta \mathbf{u}_{e,i}^n = 2\gamma \dfrac{m_e}{m_{D^e}} \Delta t \delta \Phi_{e,i}^{n+1}, \gamma \geq 1$.

Plan

Perspectives

Schéma explicite

- Stabilité.
  $\rightarrow$ Convergence des analyses (linéaire / non linéaire, semi-discret / discret / continu).
  $\rightarrow$ Ordre élevé temps et espace - Ordre très élevé (MOOD).
  $\rightarrow$ Influence de Coriolis.
- Version augmentée.

Perspectives

Schéma explicite

- Stabilité.
- Version augmentée.

Mailles décalées

- Implémentation.
- Consistance.
  $\rightarrow$ Cas MAC et Rannacher - Turek - (1D , 2D).
  $\rightarrow$ Approches DF, et variationnelle.
- Analyse asymptotique.

Perspectives

Schéma explicite

- Stabilité.
- Version augmentée.

Mailles décalées

- Implémentation.
- Consistance.
- Analyse asymptotique.

Conditions aux limites transparentes

R. Monjarret - The multi-layer shallow water model with free surface. Numerical treatment of the open boundaries (thèse - 2014).

- Validation sur le schéma explicite.
- Expérimentation dans le cas implicite.
- Implémentation sur mailles décalées.

Perspectives

Schéma explicite

- Stabilité.
- Version augmentée.

Mailles décalées

- Implémentation.
- Consistance.
- Analyse asymptotique.

Conditions aux limites transparentes

- Validation sur le schéma explicite.
- Expérimentation dans le cas implicite.
- Implémentation sur mailles décalées.

Autres perspectives opérationnelles

- Schéma hybride explicite/implicite : découplage barocline / barotrope.
- Disparition des couches, fronts secs.

Merci !