Résultats d'existence pour des problèmes elliptiques superlinéaires et résonants - trois approches

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Let $\Omega \subset \mathbb{R}^N$ be a bounded regular domain, $N \geq 3$ and $f \in L^2(\Omega)$.

Problem I :

$$\begin{cases} -\Delta u = \lambda_1 u + (u^+)^q + f(x) & \text{in } \Omega, \\ u \in H_0^1(\Omega) \end{cases} \tag{1}$$

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- $\lambda_1 =$ *first eigenvalue of* $-\Delta$ *in* $H_0^1(\Omega)$
- $q > 1$ *(superlinear problem)*

Problem (1) is **resonant and superlinear** in the sense that $g(s) = (s^+)^q$ $(q > 1)$ satisfies

$$\lim_{s \to -\infty} g(s) = 0, \quad \lim_{s \to +\infty} \frac{g(s)}{s} = +\infty,$$

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Problem II :

$$\begin{cases} -\Delta_p u = \lambda_1 |u|^{p-2} u + (u^+)^q + f(x) & \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega) \end{cases} \tag{2}$$

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Sufficient condition : If $\int_\Omega f\varphi_1 = 0$

(R) $f \in L^r(\Omega)$ for some $r > N$

and v is **the unique solution** $v \in \langle \varphi_1 \rangle^\perp$ of

$$(L) \qquad \begin{array}{rcll} -\Delta v &=& \lambda_1 v + f(x) & \text{in } \Omega, \\ v &=& 0 & \text{on } \partial\Omega \end{array}$$

$u = v + t\varphi_1$ with t s.t. $u \leq 0$ is a solution of Problem (1).

First approch : degree theory

Theorem (De Figueiredo-Cuesta-Srikanth (2003))

Let $1 < q < \frac{N+1}{N-1}$ *and* $f \in L^r(\Omega)$ *for some* $r > N$ *with*

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$p_{BT} := \frac{N+1}{N-1}$ is the "exponent of Brézis-Turner"

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- Using the Wirtinger inequality

$$(1 - \frac{\lambda_1}{\lambda_2})\|\nabla w\|_2^2 \leq C\|f\|_r \|\nabla w\|_2 + |\int_\Omega (u^+)^q w dx|.$$

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It remains to estimate $|\int_\Omega (u^+)^q w dx|$.

Hardy-Sobolev inequality :

$$\int_\Omega \frac{|v|^t}{\varphi_1^{\tau t}} \leq C\|v\|^t$$

for every $0 \leq \tau \leq 1$ and $t > 1$ such that $\frac{1}{t} = \frac{1}{2} - \frac{1-\tau}{N}$.

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$$\|w\|^2 \le C(\|f\|_{L^s})(1 + \|w\|^\delta + \|w\|)$$

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with $\delta \in]1,2[$ as $1 < q < \frac{N+1}{N-1}$.

Hence $\|w\|$ is bounded and by bootstrap $\|w\|_{C_0^1}$ also.

Second approach : lower and upper solutions

Theorem (Cuesta-C.D. (2013))

Let $1 < q < \frac{N+1}{N-1}$ *and* $f \in L^r(\Omega)$ *for* $r > N$ *such that*

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- The problem (1) has an upper solution $\beta \ll 0$.

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$$c := \inf_{\gamma\in\Gamma} \max_{s\in T_\gamma} \Phi(\gamma(s))$$

$$\Phi(u) = \frac{1}{2}\int_\Omega \left[|\nabla u(x)|^2 - \lambda_1 |u|^2\right] - \frac{1}{q+1}\int_\Omega (u^+)^{q+1} - \int_\Omega fu.$$

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If $c \in \mathbb{R}$ *and* Φ *satisfies the Palais-Smale condition, then there exists* $u \in \mathcal{C}^1_0(\overline{\Omega}) \setminus (C^\beta \cup C_\alpha)$ *solution of* (1) *with* $\Phi(u) = c$.

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First modified problem :

For $r > 1 - \min \beta$, consider the problem

$$\begin{aligned} -\Delta u = (\lambda_1 - \tfrac{1}{r} h_r(u))u + (u^+)^q + f(x) =: g_r(x,u), \qquad & \text{in } \Omega, \\ u = 0, \qquad & \text{on } \partial\Omega, \end{aligned} \tag{3}$$

where

$$\begin{aligned} h_r(u) &= 0, && \text{if } u > -r, \\ &= -(u+r), && \text{if } u \in [-r-1, -r], \\ &= 1, && \text{if } u < -r-1. \end{aligned}$$

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where

$$\begin{aligned} \gamma_r(x, u) &= u, && \text{if } u \geq \alpha_r(x), \\ &= \alpha_r(x), && \text{if } u < \alpha_r(x). \end{aligned}$$

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By the maximum principle, every solution u of (4) satisfies $u \geq \alpha_r$.

Proposition

For all $r > r_0$, $\exists u_r$ *solution of* (3) *with*

$$u_r \not\leq \beta, u_r \not\geq \alpha, u_r \geq \alpha_r \quad \text{and} \quad \overline{\Phi}_r(u_r) = c_r \tag{5}$$

where

$$c_r = \inf_{\gamma \in \Gamma} \max_{s \in T_\gamma} \overline{\Phi}_r(\gamma(s)).$$

Moreover, exists $d > 0$ *such that, for all* $r > r_0$*, we have* $c_r \leq d$.

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Claim : There exists $K > r_0$ such that, for all $r > K$, every solution u_r of (3) verifying (5) is such that $u_r > -K$.

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- $\frac{a_n}{\|u_n\|_{H_0^1}} \to -1, \quad \frac{w_n}{a_n} \overset{H_0^1}{\to} 0$ where $u_n = a_n\varphi_1 + w_n$ and $\displaystyle\int_\Omega w_n\varphi_1 = 0.$

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If we prove $\frac{w_n}{a_n} \overset{C_0^1}{\to} 0$ then, for n large enough

$$u_n = |a_n|(-\varphi_1 + \frac{w_n}{|a_n|}) \leq -\frac{|a_n|}{2}\varphi_1 \ll \beta$$

which contradicts $u_n \not\leq \beta$.

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In that case $\|w_n\|_{C^{1,\alpha}} \lesssim |a_n| + 1$ with $\alpha > 0$ and by the compact imbedding in C_0^1, up to a subsequence,

$$\frac{w_n}{a_n} \overset{C_0^1}{\to} w.$$

As $\frac{w_n}{a_n} \overset{H_0^1}{\to} 0$, we obtain $w = 0$.

w_n is solution of

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\begin{aligned}
-\Delta w_n - \lambda_1 w_n = h_n(u_n)u_n^- + (u_n^+)^q + f(x), &\quad \text{in} \quad \Omega, \\
w_n = 0, &\quad \text{on} \quad \partial\Omega, \\
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By regularity, if $h_n(u_n)u_n^- + (u_n^+)^q + f \in L^s(\Omega)$, then $w_n \in W^{2,s}(\Omega)$ and

$$\|w_n\|_{W^{2,s}} \lesssim \|h_n(u_n)u_n^- + (u_n^+)^q + f\|_{L^s} \lesssim \|(u_n^+)^q\|_{L^s} + 1. \qquad (6)$$

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Hence, we need to estimate $\|(u_n^+)^q\|_{L^s}$ pour $s > N$.

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We prove $\|u_n^+\|_{H_0^1} \leq C\|u_n\|_{H_0^1}^{1/2}$ hence, for $s_1 = \frac{2N}{N-2}$,

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Hence, we make a bootstrap, "combining" both in order to gain regularity but keeping an exponent smaller than 1.

To this aim we need the condition $q < \frac{N+1}{N-1}$.

Third approach : Nehari

Theorem (Cuesta-C.D. (2015))

For all $f \in L^2(\Omega)$ *s.t.* $\int_\Omega f\varphi_1 < 0$ *and* $1 < q \leq \frac{N+2}{N-2}$*, there exists* $\epsilon > 0$ *s.t., for all* $0 < t < \epsilon$*,*

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has at least one solution s.t. $\int_\Omega fu < 0$*.*

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$$\begin{cases} -\Delta u = \lambda_1 u + (u^+)^q + t f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least one solution s.t. $\int_\Omega fu < 0$*.*

Observe that here $f \in L^2(\Omega)$ (even $f \in L^{\frac{2N}{N+2}}(\Omega)$),
$1 < q \leq 2^* - 1 = \frac{N+2}{N-2}$!

Third approach : Nehari

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But $t\|f\|_{L^2}$ small.

The functional corresponding to the problem is

$$J(u) := \frac{1}{2}N(u) - \frac{1}{q+1}S(u) - t\,L(u)$$

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The **Nehari manifold** associated is

$$\mathcal{N} := \{u \in H_0^1(\Omega) \mid \langle J'(u), u\rangle = N(u) - S(u) - t\,L(u) = 0\}.$$

Let us set $\forall u \neq 0$, $s \geq 0$,

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We then have $\mathcal{N} = \{u \in H_0^1(\Omega) \mid j_u'(1) = 0\}$.

The maximum of J on $\mathcal{N}$

$$\begin{aligned} \mathcal{N}^- &:= \{u \in H_0^1(\Omega) \mid j_u'(1) = 0, j_u''(1) < 0\} \\ &= \{u \in \mathcal{N} \mid N(u) < q\, S(u)\}; \end{aligned}$$

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We denote

$$\mathcal{L}^- = \{u \in H_0^1(\Omega) \mid L(u) < 0\},$$

$$\mathcal{L}_0 = \{u \in H_0^1(\Omega) \mid L(u) = 0\},$$

$$\mathcal{L}_0^- := \mathcal{L}^- \cup \mathcal{L}_0.$$

Lemma (Projection on Nehari)

If $S(u) > 0$, $L(u) \leq 0$ and $L(u) < 0$ in case $N(u) = 0$, then there exists a unique $t_1 = t_1(u) > 0$ such that $t_1 u \in \mathcal{N}$. Moreover $t_1 u \in \mathcal{N}^-$ and j_u has a global maximum in t_1.

Proposition

If $q \in]1, \frac{N+2}{N-2}]$ *and* $f \in L^2(\Omega)$ *satisfies* $\int_\Omega f\varphi_1 < 0.$

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(*i*) $\inf_{\mathcal{N}\cap\mathcal{L}_0^-} J > 0$;

Proposition

If $q \in]1, \frac{N+2}{N-2}]$ *and* $f \in L^2(\Omega)$ *satisfies* $\displaystyle\int_\Omega f\varphi_1 < 0.$

Then

(i) $\inf_{\mathcal{N}\cap\mathcal{L}_0^-} J > 0\,;$

(ii) *If* $\displaystyle\inf_{u\in\mathcal{N}\cap\mathcal{L}_0^-} J(u) < \inf_{u\in\mathcal{N}\cap\mathcal{L}_0} J(u),$
then $\exists u_0 \in \mathcal{N}\cap\mathcal{L}^-$ *solution of* (1) *such that*

$$J(u_0) = \inf_{u\in\mathcal{N}\cap\mathcal{L}_0^-} J(u) = \inf_{u\in\mathcal{N}^-\cap\mathcal{L}_0^-} J(u).$$

Proof of the Theorem.

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$$\inf_{u\in\mathcal{N}\cap\mathcal{L}_0} J(u) \geq c(f). \tag{10}$$

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Hence, from (10) and (11), we will have our result if t is small enough.

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Questions : ? ? ?