



Universidad
Nacional
de Córdoba



Facultad de
Ciencias Exactas
Físicas y Naturales



NEW FORMULATION FOR CHAOTIC INTERMITTENCY WITHOUT AND WITH NOISE

SERGIO ELASKAR¹

¹Institute for Advanced Studies in Engineering and Technology, UNC-CONICET
Department of Aeronautics, UNC

Université de Pau et des Pays de l'Adour

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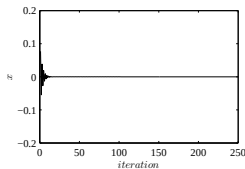


Work topics

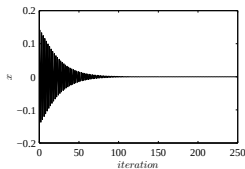
- Chaotic dynamics and intermittency.
- Numerical simulation of high enthalpy flows:
 - Chemical equilibrium gas and neutral gas (our numerical codes).
 - Without chemical and thermodynamic equilibrium.
 - OpenFOAM software.
- Numerical simulation of magnetogasdynamics flows: Our numerical codes and FLASH software. Applications to numerical simulation of solar dark lanes, loops and Moreton waves.
- Numerical simulation of wind and explosions on tanks and silos: NASTRAN, OpenFOAM and FLUENT
- Constitutive equations and finite element simulation of granular flows.
- Three wave truncation of the derivative nonlinear Schrodinger equation.
- Zero-dimensional model for pulsed plasma thrusters and some applications of tethers
- Numerical simulation of Vlasov-Poisson equations.

Chaotic intermittency

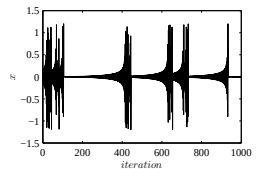
- Intermittency phenomenon is characterized by a signal that alternates randomly regular or laminar phases and irregular bursts.
- The number of chaotic bursts intensifies with an control parameter.
- Intermittency offers a continuous route from regular to chaotic motion.
- Traditionally intermittency was classified into three different types: I, II and III.
- Later studies have introduced other types of intermittencies: V, X, eyelet, off-on, ring, etc.



(a) $\varepsilon = -0.3$



(b) $\varepsilon = -0.05$



(c) $\varepsilon = 0.01$

Classical formulation of the chaotic intermittency

One dimensional map: $x_{n+1} = F(x_n)$, $F : \mathbf{R} \rightarrow \mathbf{R}$

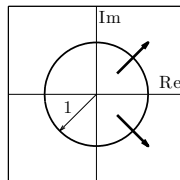
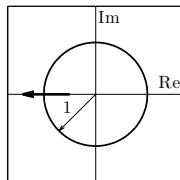
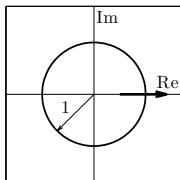
In order for chaotic intermittency can appear, two different features must exist: a local map and the reinjection mechanism.

The local maps for type-I, -II and -III intermitencies are:

Type-I intermittency: $x_{n+1} = \varepsilon + x_n + a x_n^p$ ($p = 2$)

Type-II intermittency: $x_{n+1} = (1 + \varepsilon)x_n + a x_n^p$ ($p = 3$)

Type-III intermittency: $x_{n+1} = -(1 + \varepsilon)x_n - a x_n^p$ ($p = 3$)



Classical formulation of the chaotic intermittency

Statistical parameters:

- ReInjection probability density function (RPD): $\phi(x)$
- Probability density of the laminar lengths: $\psi(x)$ or $\phi_I(x)$
- Average laminar length: $\bar{l}$
- Characteristic relation: $\bar{l} = f(\varepsilon)$

The reInjection probability density $\phi(x)$, determines the statistical distribution of the reinjected trajectories and it depends on the particular shape of $F(x)$ in the no linear region. The classical theory of intermittency establishes:

$$\phi(x) = k$$

where k is a constant value. Therefore, there is uniform reInjection.

Classical formulation of the chaotic intermittency

Laminar length also called the length of the laminar region: When a trajectory is reinjected inside the laminar zone, it evolves until it leaves the laminar interval. The number of iterations used for the trajectory to pass through the laminar interval is called the laminar length.

Type-I:

$$l(x, c) = \frac{1}{\sqrt{a\varepsilon}} \left[\arctan \left(\frac{c}{\sqrt{\varepsilon/a}} \right) - \arctan \left(\frac{x}{\sqrt{\varepsilon/a}} \right) \right]$$

Type-II:

$$l(x, c) = \frac{1}{2\varepsilon} \ln \left(\frac{a + \varepsilon/x^2}{a + \varepsilon/c^2} \right)$$

Type-III:

$$l(x, c) = \frac{1}{\varepsilon} \ln \left(\frac{a + \varepsilon/x^2}{a + \varepsilon/c^2} \right)$$

Classical formulation of the chaotic intermittency

The probability density of the laminar lengths gives the probability of finding laminar lengths between l and $l + dl$. The probability density of the laminar lengths is called $\psi(l, c)$:

$$\psi(l, c) = \phi[X(l, c)] \left| \frac{dX(l, c)}{dl} \right|,$$

where $X(l, c)$ is the inverse of $l(x, c)$.

The average laminar length $\bar{l}$ is:

$$\bar{l} = \int_0^{l_m} \psi(l, c) l(x, c) dl$$

where l_m is the highest laminar length; $l_m = l(-c, c)$.

Classical formulation of the chaotic intermittency

The **characteristic relation** establishes the relation between the average laminar length, $\bar{l}$, and the control parameter ε (for $\varepsilon \rightarrow 0$):

When $c \frac{\sqrt{a}}{\sqrt{\varepsilon}} \gg 1$ in the average laminar length equations; **the characteristic relations** result:

Type-I:

$$\bar{l} \propto \varepsilon^{-1/2}$$

Type-II:

$$\bar{l} \propto \varepsilon^{-1/2}; \quad \bar{l} \propto \varepsilon^{-1}$$

Type-III:

$$\bar{l} \propto \varepsilon^{-1/2}; \quad \bar{l} \propto \varepsilon^{-1}$$

New formulation of the chaotic intermittency

To fix ideas, let us introduce an illustrating model having type-II intermittency

$$x_{n+1} = F(x_n) \equiv \begin{cases} F_1(x_n) = (1 + \varepsilon)x + (1 - \varepsilon)x^p, & x_n < x_r \\ F_2(x_n) = (F_1(x_n) - 1)^\gamma, & x_n \geq x_r \end{cases}$$

- x_r is the root of the equation $F_1(x_r) = 1$. F_1 drives the laminar dynamics whereas F_2 drives the reinjection mechanism from the chaotic region into the laminar region.
- For $p = 2$ and $\gamma = 1$, the seminar paper of Manneville is recovered.
- $x = 0$ is a fixed point of F_1 . It is stable for $-2 < \varepsilon < 0$ and it is unstable for $\varepsilon > 0$.
- The iterated points x_n of a starting point x_1 closed to the origin, increases in a process driven by parameters ε and p . When x_n becomes larger than x_r , a chaotic burst occurs that will be interrupted when x_n is again mapped into the laminar region by means of $F_2(x)$ which determines the RPD.

New formulation of the chaotic intermittency

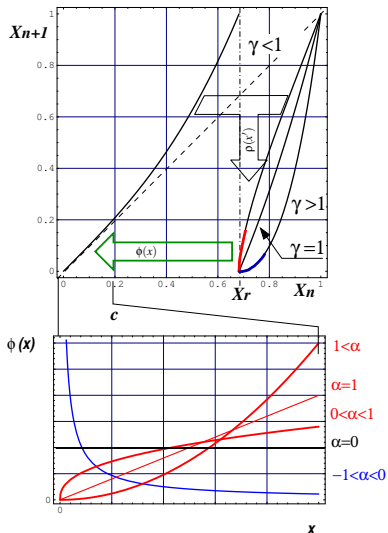
The reinjection mechanism $F_2(x)$ for $\gamma = 1$ produces uniform reinjection. For $\gamma \neq 1$, $\phi(x)$ is not uniform.

All points reinjected in the laminar region, $[0, c]$, are coming from the points close to x_r and $x_i \gtrsim x_r$.

$$\lambda \rho(x_i) dx_i = \phi(x) F_2'(x_i) dx_i$$

$$\phi(x) = \rho(x_i) \frac{\lambda}{\left. \frac{dF_2(\tau)}{d\tau} \right|_{\tau=x_i}},$$

where λ is a normalization constant such that $\int_0^c \phi(\tau) d\tau = 1$.



New formulation of the chaotic intermittency

Remember that

$$F_2(x_i) = (F_1(x_i) - 1)^\gamma$$

Then

$$\phi(x) = \rho(x_i) \frac{\lambda}{\left. \frac{dF_2(\tau)}{d\tau} \right|_{\tau=x_i}} = \frac{K \rho(x_i)}{\gamma F_1'(x_i)} x^{\frac{1}{\gamma}-1}$$

where $F_1'(x)$ indicates the derivative of the function $F_1(x)$.

- In the linear approximation of $F_1(x)$ in the interval $(x_r, F_2^{-1}(c))$, we can consider $F_1'(x)$ as a constant.
- If the density $\rho(x_i)$ is uniform, we get the follow reinjection probability density:

$$\phi(x) = b x^\alpha \quad \text{where} \quad \alpha = \frac{1}{\gamma} - 1.$$

where b is a normalization constant given by $\int_0^c \phi(\tau) d\tau = 1$.

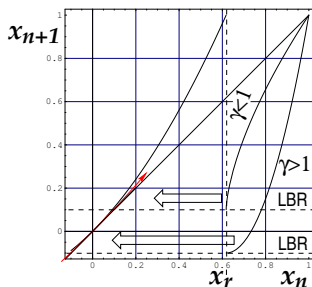
- The function PRD will strongly depend on parameter γ , that determines the curvature of the map in region marked by colored segments (close to x_r).

New formulation of the chaotic intermittency

There are reinjection mechanisms only reinject points for values of $x \geq \hat{x}$. Then, $\phi(x) = 0$ for $x < \hat{x}$. This threshold value is referred as **lower boundary reinjection point (LBR)**. To consider the effect of $LBR \neq 0$ we propose the following map having type-I intermittency:

$$x_{n+1} = G(x_n) = \begin{cases} \varepsilon + x_n + a x_n^p & \text{if } x_n < x_r \\ (1 - \hat{x}) \left(\frac{x_n - x_r}{1 - x_r} \right)^\gamma + \hat{x} & \text{otherwise} \end{cases}$$

x_r is the root of the equation $\varepsilon + x_n + x_n^p = 1$. γ drives the nonlinear term of the reinjection mechanism. $\hat{x}$ is the lower boundary reinjection point (LBR).



New formulation of the chaotic intermittency

For $\hat{x} \neq 0$, a similar argument proposed by type-II intermittency provides the same power law, but now including $\hat{x}$ as following:

$$\phi(x) = b (x - \hat{x})^\alpha$$

- The new RPD includes the classical approach as the particular case: $\alpha = 0$.
- The free parameters $\hat{x}$ and α are determined by the dynamics in the chaotic region.
- The parameter $\hat{x}$ corresponds with the lower bound of reinjection (LBR).
- The exponent α is generated by the trajectories within chaotic regime in a vicinity of a point in the map with infinite or zero tangent.
- b is a normalization constant.

The function $M(x)$

The RPD function determines the statistical distribution of trajectories leaving chaotic region going back into the laminar region. As the RPD is in general the power law the key point to solve the problem of model-fitting is to introduce the following integral characteristic:

$$M(x) = \begin{cases} \frac{\int_{x_t}^x \tau \phi(\tau) d\tau}{\int_{x_t}^x \phi(\tau) d\tau} & \text{if } \int_{x_t}^x \phi(\tau) d\tau \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

where x_t is some “starting” point.

- The interesting property of the function $M(x)$ is that it is a linear function for the power law RPD
- Hence the function $M(x)$ is an useful tool to find the parameters $\hat{x}$ and α determining the RPD.
- As $M(x)$ is an integral characteristic, its numerical estimation is more robust than direct evaluation of $\phi(x)$. This allows reducing statistical fluctuations

Fitting linear model to experimental data

We notice that $M(x)$ is an average over reinjection points in the interval (x_t, x) , hence we can write

$$M(x) = M_j \equiv \frac{1}{j} \sum_{k=1}^j x_k, \quad x_{j-1} < x \leq x_j$$

where the data set (N reinjection points) $\{x_j\}_{j=1}^N$ has been previously ordered, i.e. $x_j \leq x_{j+1}$.

For a linear function $M(x)$

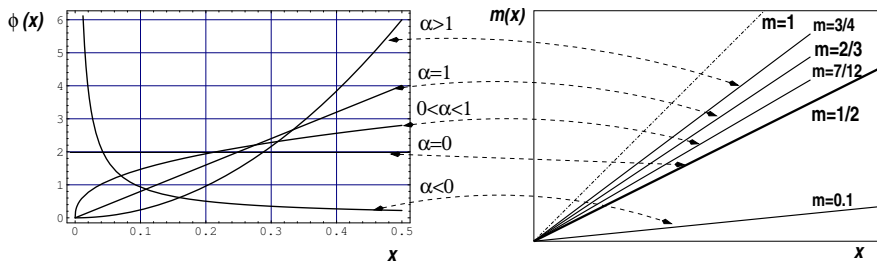
$$M(x) = \begin{cases} m(x - \hat{x}) + \hat{x} & \text{if } x \geq \hat{x} \\ 0 & \text{otherwise} \end{cases}$$

where $m \in (0, 1)$ is a free parameter and $\hat{x}$ can be approximated by $\hat{x} \approx \inf\{x_j\}$.

The RPD results

$$\phi(x) = b (x - \hat{x})^\alpha, \quad \alpha = \frac{2m - 1}{1 - m}$$

Fitting linear model to experimental data



This figure displays different RPD depending on the exponent α for $\hat{x} = 0$ and $c = 0.5$.

It is also shown how the free parameter α depends on the slope m

For $m = 1/2$, we obtain $\alpha = 0$, and $\phi(x) = \text{constant}$

Type-II intermittency

The probability density of the length of laminar phase $\psi(l)$

$$\psi(l) = \phi(X(l)) \left| \frac{dX(l)}{dl} \right| = \phi(X(l)) |\varepsilon X(l) + a X(l)^p|$$

where $X(l)$ is the inverse function of $l(x, c)$

$\psi(l)$ depends on the RPD, which is an important difference with the classical theory

Finally, by using the previous equations and after some algebraic manipulation we get

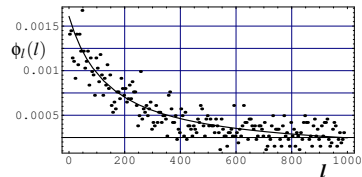
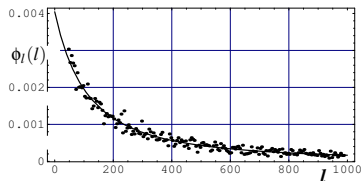
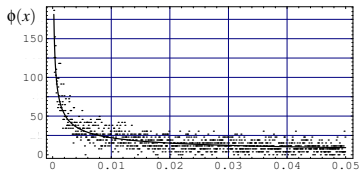
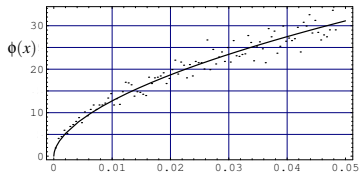
$$\psi(l) = b \left(\left(\frac{\varepsilon}{\left(a + \frac{\varepsilon}{c^{(p-1)}} \right) e^{(p-1)\varepsilon l} - a} \right)^{\frac{1}{p-1}} - \hat{x} \right)^{\alpha} \times$$

$$\left(\frac{\varepsilon}{\left(a + \frac{\varepsilon}{c^{(p-1)}} \right) e^{(p-1)\varepsilon l} - a} \right)^{\frac{p}{p-1}} \left(a + \frac{\varepsilon}{c^{(p-1)}} \right) e^{(p-1)\varepsilon l}$$

for $\hat{x} = 0$:
$$\psi(l) = b \left(\frac{\varepsilon}{\left(a + \frac{\varepsilon}{c^{(p-1)}} \right) e^{(p-1)\varepsilon l} - a} \right)^{\frac{p+\alpha}{p-1}} \left(a + \frac{\varepsilon}{c^{(p-1)}} \right) e^{(p-1)\varepsilon l}$$

It depends on α , which is determined by the slope m of the linear function $M(x)$.

Examples - Results



Upper figures: $\gamma = 2$ and $\epsilon = 10^{-3} \rightarrow m \simeq 0.32$. Lower figures: $\gamma = 0.65$ and $\epsilon = 10^{-4} \rightarrow m \simeq 0.61$.

Type-II intermittency

The average laminar length $\bar{l}$ given by

$$\bar{l} = \int_0^\infty s\psi(s)ds = \int_{\hat{x}}^c l(x, c)\phi(x)dx.$$

This integral depends on the LBR. Let us start our analysis with $\hat{x} = 0$.

For small values of ε :

$$\bar{l} \approx \frac{1}{ac^{\alpha+1}} \left(\frac{a}{\varepsilon}\right)^{\frac{p-\alpha-2}{p-1}} \frac{\pi}{p-1} \sin^{-1} \left(\frac{\pi(1+\alpha)}{p-1} \right)$$

Assuming that α remains constant as ε changes, [the characteristic relation](#) yields

$$\bar{l} \propto \varepsilon^{-\beta}, \quad \beta = \frac{p-\alpha-2}{p-1} = \frac{p(1-m)-1}{(p-1)(1-m)}$$

Type-II intermittency

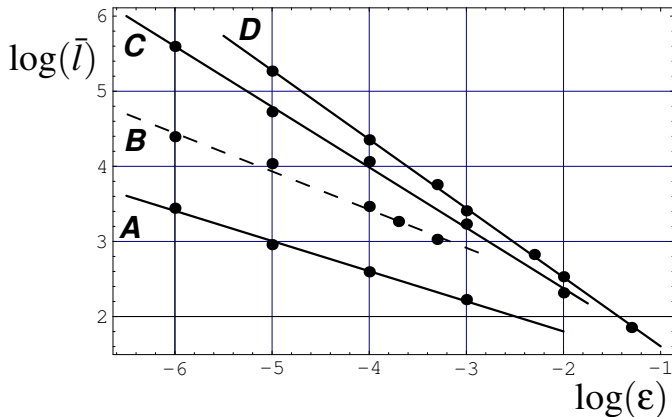


Figure : Characteristic relation for different values of the parameter γ . Dots show numerical data and lines show the best fit straight. For lines B,C and D we used $P = 3$ and the values of γ are 1, 2 and 3 for lines B, C and D respectively. The value of parameters for line A are $P = 2$ and $\gamma = 1.5$.

Type-III intermittency

A typical map exhibiting type-III intermittency is (Elaskar and del Río, 2011):

$$x_{n+1} = F_{III}(x_n) = -(1 + \varepsilon) x_n - a x_n^3 + d x_n^6 \sin(x_n)$$

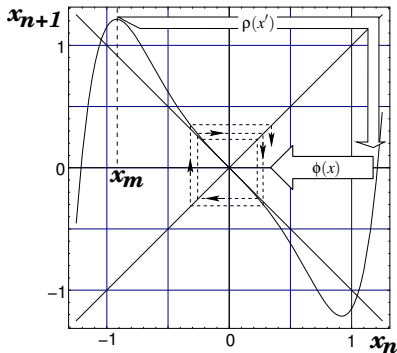


Figure : Maps illustrating type-III intermittencies. The reinjection mechanism is indicated by empty arrows. Dashed arrow illustrates the trajectory inside of laminar region, and x_m indicates the maximum of the map.

Type-III intermittency

This map presents two main differences with respect the previous one. Firstly, points in the neighbor of the extreme point need two map iterations to reach the laminar reinjection and secondly it has a symmetric reinjection mechanism into the laminar region.

As a consequence of the indirect reinjection mechanism

$$\rho(x') \propto \rho(x'') \frac{1}{\left. \frac{dF(\tau)}{d\tau} \right|_{\tau=x''}}$$

where x'' needs two iterations on the map to reach the laminar region. In a second step we connect $\rho(x')$ and $\phi(x)$ by

$$\phi(x) = \rho(x') \frac{C}{\left. \frac{dF(\tau)}{d\tau} \right|_{\tau=x'}}$$

where C is a suitable constant.

Type-III intermittency

As $\frac{dF(x')}{dx'}$ is non-zero and bounded, by means of the approximation

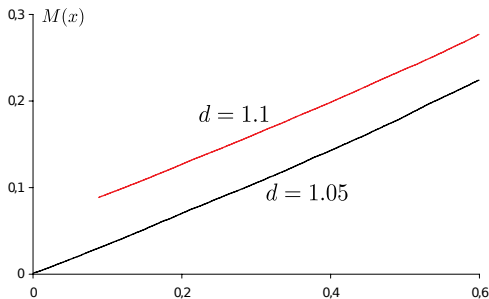
$$K \approx \frac{C}{\left. \frac{dF(\tau)}{d\tau} \right|_{\tau=x'}}. \quad (1)$$

We recover an equation like but now referring to $\rho(x'')$, that is, two steps back from the laminar region

$$\phi(x) = K \frac{\rho(x'')}{\left. \frac{dF(\tau)}{d\tau} \right|_{\tau=x''}}.$$

Notice that, to be reinjected into the laminar region in two iterations, x'' must be close to x_m

Examples - Results

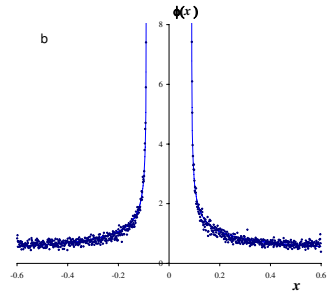
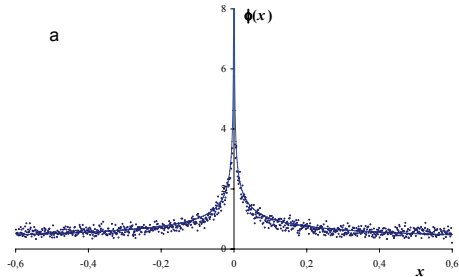


Numerical function $M(x)$ for two sets of parameters. Lower line $d = 1.05$, $a = 1$ and $\varepsilon = 0.01$. Upper line $d = 1.1$, $a = 1$ and $\varepsilon = 0.01$.

After numerical fitting, we have the two parameters to determine the RPD: $m \approx 0.36$, $\hat{x} \approx 0$ and $m \approx 0.37$, $\hat{x} \approx 0.053$, respectively.

Examples - Results

Left: $m \approx 0.36$, $\hat{x} \approx 0$. Right: $m \approx 0.37$, $\hat{x} \approx 0.053$.



Type-III intermittency

The map of the figure has two symmetric reinjection mechanisms. There are two LBR: $\hat{x} > 0$ and $\hat{x} < 0$. For this last case the RPD must take into account the overlapping of the two symmetry reinjections and should be described by the following function:

$$\phi(x) = \begin{cases} b[(|\hat{x}| + x)^\alpha + (|\hat{x}| - x)^\alpha] & \text{if } |x| \leq |\hat{x}| \\ b(|\hat{x}| + x)^\alpha & \text{if } |\hat{x}| < x \leq c \\ b(|\hat{x}| - x)^\alpha & \text{if } -c < x \leq -|\hat{x}| \end{cases}$$

where $b > 0$ is again obtained by the standard normalization condition.

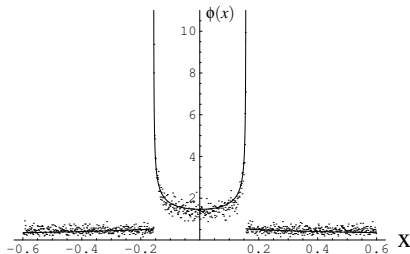


Figure : RPD for $LBR < 0$. $\hat{x} = x_i \simeq -0.157$. $a = 1.035$, $b = 1.05$, $\varepsilon = 0.001$.

Type-III intermittency

For $\hat{x} < 0$, $\phi(x)$ is specified by two parameters: α and $\hat{x}$. Now $M(x)$ is not linear. But, it is still useful to determine the RPD. $\phi(x)$ is non-continuous for $x = |\hat{x}|$, hence $M(x)$ is non-differentiable at this point. Then, $\hat{x}$ must appear as a vertex point for $M(x)$. To find α we evaluate $M(x)$ using its definition (for $0 < x \leq |\hat{x}|$):

$$M(x) = \frac{1}{(2+\alpha)} \left[(1+\alpha)x - |\hat{x}| + 2 \frac{|\hat{x}| (|\hat{x}| - x)^{1+\alpha} - |\hat{x}|^{2+\alpha}}{(|\hat{x}| - x)^{1+\alpha} - (|\hat{x}| + x)^{1+\alpha}} \right]$$

for $x = |\hat{x}|$ we have: $M(|\hat{x}|) = \frac{\alpha+2^{-\alpha}}{\alpha+2} |\hat{x}|$ from which α can be obtained.

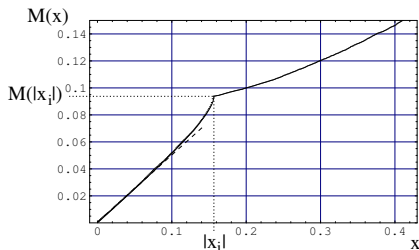
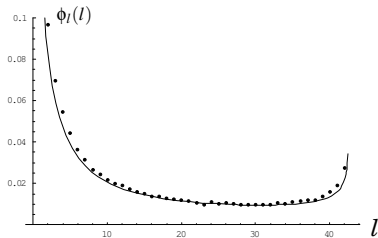
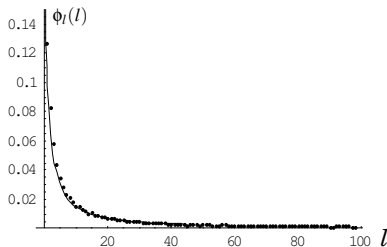


Figure : Numerical $M(x)$. $\hat{x} = x_i \simeq -0.157$. $a = 1.035$, $b = 1.05$, $\varepsilon = 0.001$.

Probability density of the laminar lengths - Results

Left: $m \approx 0.36$, $\hat{x} \approx 0$. Right: $m \approx 0.37$, $\hat{x} \approx 0.053$.



Probability density of the laminar lengths - Results

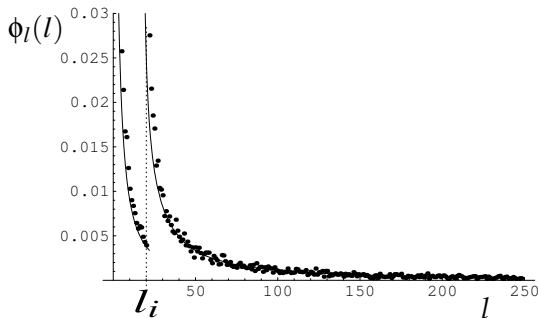


Figure : $\psi(l)$ for $LBR < 0$. $\hat{x} = x_i \simeq -0.157$. $a = 1.035$, $b = 1.05$, $\varepsilon = 0.001$.

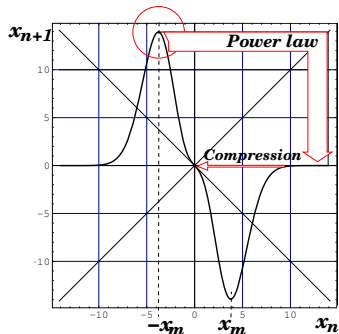
Application of the new formulation to pathological cases

Laugessen map (1997):

$$x_{n+1} = F(x_n) = -x_n (1 + \varepsilon + x_n^2) e^{-dx_n^2}$$

d is a constant. For $\varepsilon > 0$ the map has a single unstable fixed point at $x_0 = 0$.

The behavior of trajectories near x_0 defines the laminar phase of intermittency. The local map in the neighborhood of $x = 0$ is determined by the Taylor series expansion of the map as following: $x_{n+1} = -(1 + \varepsilon) x_n - (1 - (1 + \varepsilon) d) x_n^3$.



Application of the new formulation to pathological cases

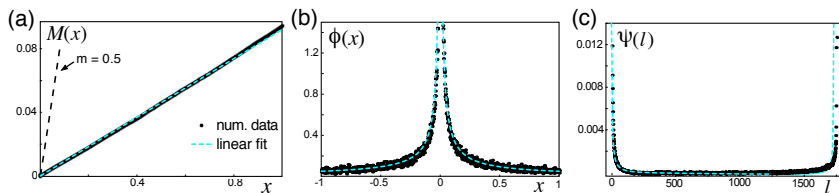


Figure : Analysis of the anomalous Laugesen type-III intermittency: $d = 0.1$, $\varepsilon = 0.005$, and the laminar interval $[-1, 1]$. a) Assessment of the RPD by numerical simulation. Dots correspond to $M(x)$. Cyan dashed line corresponds to the least squares fit. Black dashed line with slope $m = 0.5$ corresponds to the uniform RPD. b) Numerical RPD. Dashed cyan curve corresponds to theoretical evaluation. c) Probability density of the length of the laminar phase. Theoretical evaluation: Cyan dashed line.

Application of the new formulation to pathological cases

Pikovsky's map has multichannel reinjection and it is an example of nonstandard intermittency. It is defined as follow (1983):

$$x_{n+1} = F_p(x_n) = \begin{cases} G(x_n) & x_n \geq 0 \\ -G(-x_n) & x_n < 0 \end{cases}$$

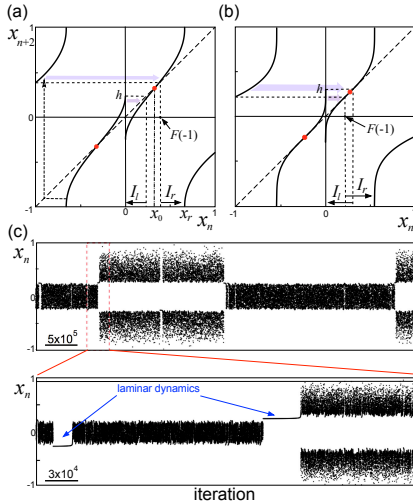
where $G(x) = x^q + hx - 1$ ($q, h > 0$). This map has no fixed points but it has a stable period-2 cycle, hence, to facilitate the study of its dynamics it is convenient to introduce the second iteration, i.e. to consider:

$$F(x) = F_p^2(x) = F_p(F_p(x))$$

which has two fixed points. In what follows we shall deal with this new map.

The local map for type-III intermittency is $x_{n+1} = -(1 + \varepsilon)x_n - a x_n^3$. Using a Taylor series expansion, we have that $a = F'''(x_0)/6$ and $\varepsilon = F'(x_0) - 1$

Application of the new formulation to pathological cases



Second iteration of the map demonstrating the Pikovsky type-II intermittency. a) Non-overlapping case with a gap between two reinjection intervals. Arrows show two routs of reinjection into two disjoint intervals I_l and I_r for the upper laminar region. Red dots mark positions of the fixed points. There are two chaotic attractors in the map. b) Slightly overlapping case. Reinjection intervals I_l and I_r overlap. There exists single chaotic attractor. c) Time evolution of the map corresponding to the case (b). Bottom subplot shows zoomed trajectory with two laminar phases near two unstable fixed points ($h = 0.255$, $q = 0.29$).

Application of the new formulation to pathological cases

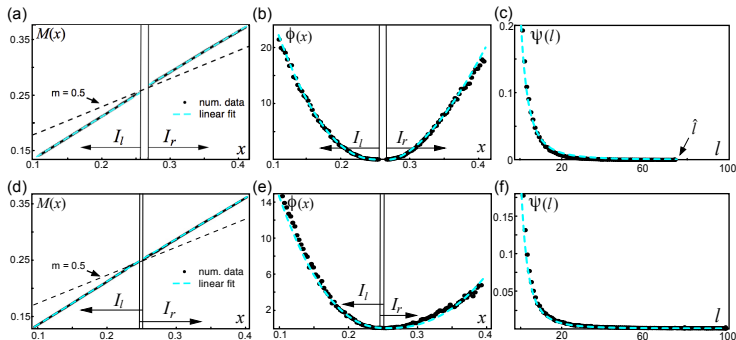


Figure : Analysis of the Pikovsky intermittency in the non-overlapping (top row, $q = 0.29$, $h = 0.255$, two chaotic attractors) and slightly overlapping (bottom row, $q = 0.27$, $h = 0.255$, single chaotic attractor) cases. Results are shown for the second iteration of the map, d) Numerical data (dots) for two branches of $M(x)$ for reinjections in the intervals I_l and I_r . Dashed cyan lines show the corresponding least mean square fits, which then are used to plot $\phi(x)$ and $\psi(l)$. Dashed line with slope $m = 0.5$ corresponds to the uniform RPD.

Noise effects in chaotic intermittency

- In previous studies about noise, only the local effect was assumed and the noise strength σ is much smaller than the control parameter ϵ . These studies were carried out by means of the renormalisation group analysis or by using the Fokker-Plank equation. There are not studies focused on the effect of noise on the RPD as far as the author knows.
- Since the noise is always present in nature, it is of a fundamental importance to know the effect of noise on the intermittency phenomenon.

We introduce an illustrating example having type-II intermittency:

$$x'_{n+1} = \begin{cases} F(x_n) + \sigma \xi_n & x_n \leq x_r \\ (F(x_n) - 1)^\gamma + \sigma \xi_n & x_n > x_r, \end{cases}$$

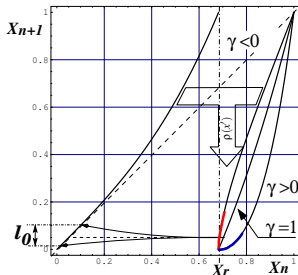
where ξ_n is an uniform distributed noise verifying that $\langle \xi_m, \xi_n \rangle = \delta(m - n)$ and $\langle \xi_n \rangle = 0$. The noise strength is given by σ , and $F(x) = (1 + \epsilon)x_n + (1 - \epsilon)x_n^3$. As before, we denote by x_r the root of the equation $F(x_r) = 1$.

Noisy reinjection probability density function (NRPD)

To get an analytical expression for the **noisy reinjection probability density function (NRPD)**, denoted here $\Phi(x)$, whereas we reserve $\phi(x)$ for the noiseless RPD, we analyze the effect of noise on the reinjection trajectories. A noiseless trajectory represented by a dashed line, is perturbed by noise. As a consequence of this, the reinjection point must end inside of an interval represented in the figure by l_0 . That is, the noiseless density $\phi(x)$ should be transformed into a new density $\Phi(x)$ according to the convolution:

$$\Phi(x) = \int \phi(y)G(x-y, \sigma)dy,$$

where $G(x, \sigma)$ is the probability density of the noise term $\sigma\xi_n$.



Noisy reinjection probability density function (NRPD)

Expanding $\phi(x)$ at to the first term:

$$\Phi(x) \approx \int \phi(x) G(x-y, \sigma) dy + \int \frac{d\phi(x)}{dx} (x-y) G(x-y, \sigma) dy.$$

Taking into account that $\int G(x, \sigma) dx = 1$, we have

$$\Phi(x) \approx \phi(x) + \frac{d\phi(x)}{dx} \int (x-y) G(x-y, \sigma) dy$$

so we expect $\Phi(x) \approx \phi(x)$ in regions where the slope of $\phi(x)$ is small enough.

Noise effects on type-II intermittency

$$x_{n+1}^1 = \begin{cases} F(x_n) + \sigma \xi_n & x_n \leq x_r \\ (F(x_n) - 1)^\gamma + \sigma \xi_n & x_n > x_r, \end{cases}$$

Because of the noise, the value of x'_{n+1} may be mapped out of the unit interval, so to keep x_{n+1}^1 in the unit interval we define the following map

$$x_{n+1} = \begin{cases} |x_{n+1}^1| & x_{n+1}^1 \leq 1 \\ |x_{n+1}^1| - 2 \bmod(|x_{n+1}^1|, 1) & x_{n+1}^1 > 1. \end{cases}$$

Note that $x_{n+1} = x_{n+1}^1$ for $x_{n+1}^1 \in [0, 1]$.

We have $\hat{x} = 0$, hence $\phi(x) = b x^\alpha$. The function $G(x, \sigma)$ can be written as:

$$G(x, \sigma) = \frac{\Theta(x + \sigma) - \Theta(x - \sigma)}{2\sigma},$$

where $\Theta(x)$ is the Heaviside step function.

Then, the NRPD results as:

$$\Phi(x) = \frac{1}{c^{1+\alpha}} \frac{(x + \sigma)^{1+\alpha} - \text{Sg}(x - \sigma)|x - \sigma|^{1+\alpha}}{2\sigma}$$

Where we denote by $\text{Sg}(x)$ the sign function that extracts the sign from its argument.

Noise effects on type-II intermittency

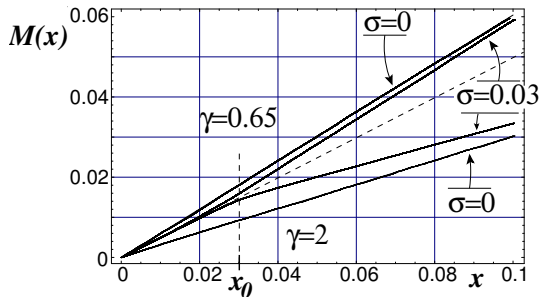


Figure : Function $M(x)$. The dashed line with slope 1/2 shows the uniform reinjection case. The lines above the dashed one correspond to $\gamma = 0.65$ for two values of the noise strength as indicated. The same values of noise strength are used for the two lines below the dashed one, that correspond to $\gamma = 2$. For all the cases $\varepsilon = 0.001$.

Noise effects on type-II intermittency

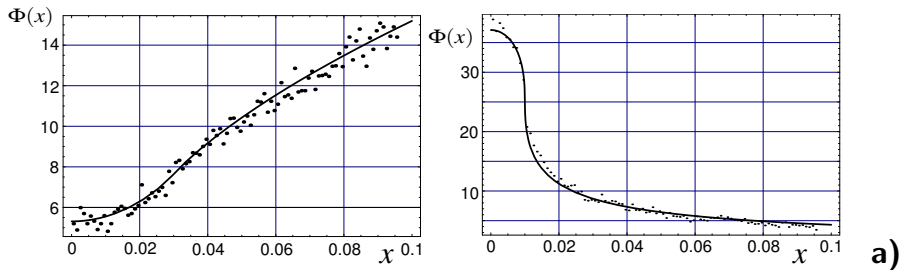
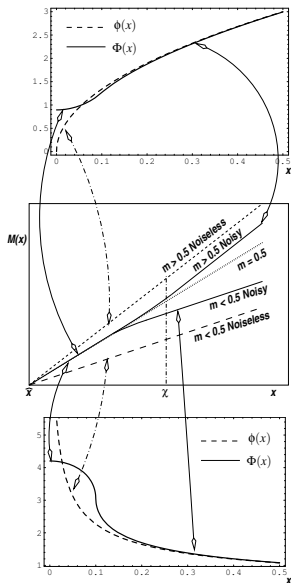


Figure : NRPD using two sets of parameters: a) $c = 0.1$, $\gamma = 0.65$, $\sigma = 0.03$ and b) $c = 0.1$, $\gamma = 2$ and $\sigma = 0.01$. Dots correspond to numerical data whereas the analytical approximation is plotted as a solid line.

Noise effects on type-II intermittency



Noise effects on type-III intermittency

We consider the following map formed by composition of the following noiseless maps

$$x_n^* = F(x_n) = -(1 + \varepsilon) x_n - a x_n^3 + d x_n^6 \sin(x_n)$$

and the random perturbed map defined as

$$x_{n+1} = x_n^* + \sigma \xi_n,$$

where we just add a noise to the variable.

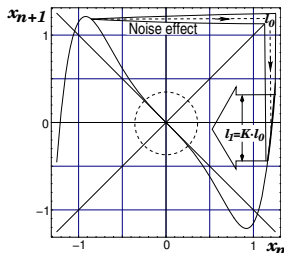


Figure : Dashed line between the two solid lines indicate the effect of the map on a point near the maximum. These solid lines indicate the effect of the noisy map on the same point, that will be mapped on the interval I_1 on the graph of the map.

Noise effects on type-III intermittency

We consider two maps:

Map presented by Elaskar and del Rio (2011)

$$x_{n+1} = F(x_n) = -(1 + \varepsilon)x_n - ax_n^3 + dx_n^3 \sin(x_n) + \sigma \xi_n$$

For this map the derivative $K = dF(x_n)/dx \simeq 11$. Where $x_{n+1} = F(x_n)$ is a reinjected point into the laminar zone.

Map presented by Laugesen et al. (1997)

$$x_{n+1} = F(x_n) = -((1 + \varepsilon)x_n + x_n^3)e^{(-bx_n^2)} + \sigma \xi_n$$

For this map the derivative $K = dF(x_n)/dx \simeq 0$.

ξ_n is a uniform distributed noise verifying that $\langle \xi_m, \xi_n \rangle = \delta(m - n)$ and $\langle \xi_n \rangle = 0$, with noise strength σ .

Noise effects on type-III intermittency

If the slope $K \neq 0$ and K does not reach a high value, it influences the noisy reinjection process, resulting in a more complex RPD structure. Thus, we need to consider the two iteration process until the reinjection to obtain an analytical expression for the NRPD:

$$x_n^* = F(x_n) = -((1 + \varepsilon)x_n + x_n^3)e^{-bx_n^2} \quad (2)$$

and the map defined as

$$x_{n+1} = R(x_n^*) = x_n^* + \sigma \xi_n \quad (3)$$

The density function produced by the map (3) can be calculated using the following convolution integral:

$$\rho(x) = \int \rho^*(t) G(x - t, \sigma) dt \quad (4)$$

where $\rho^*(t)$ is the density produced by the noiseless map given by Eq. (2).

Noise effects on type-III intermittency

If we consider points x_n placed close to the maximum (or minimum) of the map (2), the points x_{n+1} are not directly mapped inside of the laminar interval. It is necessary to consider other iteration to reach the laminar interval:

$$x_{n+1}^* = F(x_{n+1}) = -((1 + \varepsilon)x_{n+1} + x_{n+1}^3)e^{(-bx_{n+1}^2)} \quad (5)$$

and

$$x_{n+2} = R(x_{n+1}^*) = x_{n+1}^* + \sigma \xi_{n+1} \quad (6)$$

The resulting density function, applying Eq. (5), can be written as:

$$\rho_I(x) = \frac{dF^{-1}(x)}{dx} \rho(F^{-1}(x)) = K \rho(F^{-1}(x)) \quad (7)$$

Noise effects on type-III intermittency

Finally, to obtain the noisy reinjection probability density function, $\Phi(x)$, it is necessary to apply a new convolution integral as follow:

$$\Phi(x) = \int \rho_I(y) G(x - y, \sigma) dy \cong \int \rho(y) G(x - y, K\sigma) dy \quad (8)$$

From the previous relations (4) and (8), we note that the amplified noise intensity is given by $(K + 1)\sigma$ and the NRPD will have the exponent $\alpha + 2$, where α is the exponent for the RPD without noise.

$$\Phi(x) = \frac{\int [(y + \sigma - \hat{x})^{(1+\alpha)} - H(y - \sigma - \hat{x})(y - \sigma - \hat{x})^{(1+\alpha)}] G(x - y, K\sigma) dy}{2\sigma(c - \hat{x})^{(1+\alpha)}} \quad (9)$$

The amplified noise intensity is: $(K + 1)\sigma$.

Noise effects on type-III intermittency

NRPD without AR and amplified noise intensity $< \text{LBR}$. When $(K + 1)\sigma < \hat{x}$, the LBR changes by the noise effect. A “new LBR” arises depending on the original LBR and the amplified noise strength, defined as the difference $x_i = \hat{x} - (K + 1)\sigma$. We use the convolution integral (9) to evaluate the NRPD (Φ). Where $H(x)$ is the Heaviside function, and the expression inside the square brackets is the density, $\rho(x)$, after the first iteration for points close to the local maximum or minimum, x_m (see Eq. (4)). According to the convolution integral the NRPD depends on the value of the slope K , and three cases can appear: $0 < K \leq 1$; $1 \leq K \leq 2$; and $2 \leq K$.

NRPD without AR and amplified noise intensity $\geq \text{LBR}$: $(K + 1)\sigma \geq \hat{x}$ and $\sigma < \hat{x}$. The condition $(K + 1)\sigma \geq \hat{x}$ implies that the “new LBR” is $x_i = 0$ and reinjection trajectories coming from points $x < 0$ will occur. The restriction $\sigma < \hat{x}$ implies that the “new LBR” is the origin only when a two iteration reinjection process is considered for trajectories coming from points close to x_m . Then, σ alone is not enough to produce the reinjection from $x < 0$. Also, the function Φ depends on the value of the slope K . Therefore, three cases can appear: $0 < K \leq 1$; $1 \leq K \leq 2$; and $2 \leq K$. To obtain the function Φ we again calculate the convolution integral (9).

Noise effects on type-III intermittency

- NRPD for $0 < K \leq 1$:

For $\hat{x} - (K + 1)\sigma \leq x \leq \hat{x} + (K - 1)\sigma$

$$\Phi(x) = \Phi_1(x) = \frac{(x + \sigma(1 + K) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)}\sigma^2 K(\alpha + 1)(\alpha + 2)}$$

For $\hat{x} + (K - 1)\sigma \leq x \leq \hat{x} + (1 - K)\sigma$

$$\Phi(x) = \Phi_2(x) = \frac{(x + \sigma(1 + K) - \hat{x})^{2+\alpha} - (x + \sigma(1 - K) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)}\sigma^2 K(\alpha + 1)(\alpha + 2)}$$

For $\hat{x} + (1 - K)\sigma \leq x \leq \hat{x} + (1 + K)\sigma$

$$\Phi(x) = \frac{(x + \sigma(1 + K) - \hat{x})^{2+\alpha} - (x + \sigma(1 - K) - \hat{x})^{2+\alpha} - (x + \sigma(K - 1) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)}\sigma^2 K(\alpha + 1)(\alpha + 2)}$$

For $\hat{x} + (1 + K)\sigma \leq x$

$$\Phi(x) = \frac{(x + \sigma(1 + K) - \hat{x})^{2+\alpha} - (x + \sigma(1 - K) - \hat{x})^{2+\alpha} - (x + \sigma(K - 1) - \hat{x})^{2+\alpha}}{4(c - \hat{x})^{(\alpha+1)}\sigma^2 K(\alpha + 1)(\alpha + 2)}$$

Noise effects on type-III intermittency

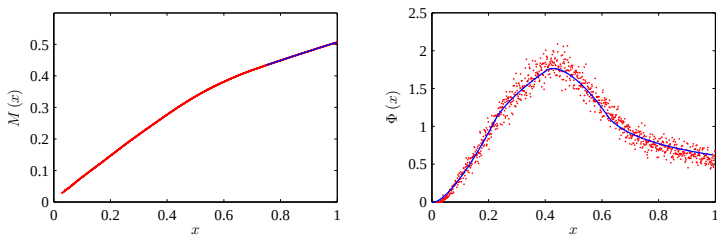


Figure : $(K + 1)\sigma < \hat{x}$ and $0 < K \leq 1$. Left: $M(x)$ function, the darker (blue) line shows the data used to calculate the exponent α . Right: NRPD function, numerical data and theoretical result are represented by points and line respectively. Parameters: $c = 1$, $\varepsilon = 0.005$, $\sigma = 0.2$, $\hat{x} \cong 0.3181$, $K \cong 0.5366$.

Noise effects on type-III intermittency

If there is no LBR. The NRPD for Laugesen map is given by

$$\Phi(x) = \frac{1}{c^{1+\alpha}} \frac{(|x| + \sigma)^{1+\alpha} - \text{Sg}(|x| - \sigma) |x| - \sigma|^{1+\alpha}}{2\sigma}$$

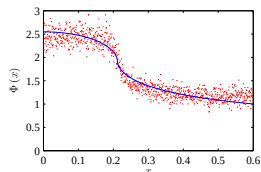
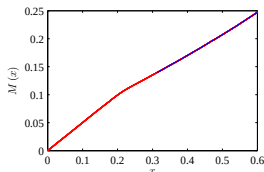
This equation is simplified because $K \simeq 0$.

On the other hand for the map given by Elaskar and del Rio, the NRPD function results:

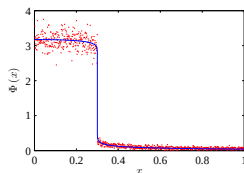
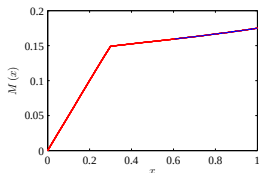
$$\Phi(x) = \frac{1}{c^{1+\alpha}} \frac{(|x| + K\sigma)^{1+\alpha} - \text{Sg}(|x| - K\sigma) |x| - K\sigma|^{1+\alpha}}{2K\sigma}$$

This equation is simplified because $K \gg 1$

Noise effects on type-III intermittency



Elaskar and del Rio map. Parameters: $\varepsilon = 0.0005$, $c = 0.6$, $a = 1.1$ and $b = 1.35$, $K = 10.5833$.



Laugesen map. Parameters: $\varepsilon = 0.005$, $c = 1.0$, $\sigma = 0.3$, $b = 0.05$, $K \cong 0$.

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Characteristic relations

Let us describe how the **characteristic exponent** is affected by the RPD. As the exponent β is defined by the characteristic relation, we need to evaluate the **average laminar length**

Taking into account the function ψ , depending on the parameter $\hat{x}$ and α , a similar analysis that had been carried out for type-II intermittency provides the characteristic exponent β for type-I intermittency. It is not a single value:

- Case D: $\hat{x} \approx x_0$

- D1: $m \in (0, 1 - \frac{1}{p})$ or equivalent $\alpha \in (-1, p - 2)$.

$$\beta = \frac{p - \alpha - 2}{p} = 1 - \frac{1}{(1 - m)p}.$$

Particularly $\lim_{m \rightarrow 0} \beta = 1 - \frac{1}{p}$ and $\lim_{m \rightarrow 1 - \frac{1}{p}} \beta = 0$.

- D2: $m \in [1 - \frac{1}{p}, 1)$ or equivalent $p - 2 < \alpha$. Now we have: $\beta = 0$

- Case E: $\hat{x} > x_0$: $\beta = 0$

- Case F: $\hat{x} < x_0$.

- F1: $\alpha > 0$: $\beta = \frac{p-2}{p}$.
- F2: $\alpha < 0$: $\beta = \frac{p-1}{p}$.

Noise effects on type-III intermittency

The noise effect can be obtained according to the convolution

$$\rho(x) = \int \rho'(\tau) G(\tau - x, \sigma) d\tau,$$

$\rho'(x)$ is the invariant density of the map, and $\rho(x)$ is the invariant density observed in the interval I_0 . On the contrary of type-II case, points on I_0 are not directly mapped on the laminar region, so to get the NRPD we must propagate the density $\rho(x)$

$$\rho_I(x) = \frac{dF^{-1}(x)}{dx} \rho(F^{-1}(x)).$$

Finally, to include the noise effect into the density $\rho_I(x)$. The NRPD is given by

$$\Phi(x) = \int \rho_I(y) G(y - x, \sigma) dy.$$

$$\Phi(x) = \frac{1}{c^{1+\alpha}} \frac{(|x| + K\sigma)^{1+\alpha} - \text{Sg}(|x| - K\sigma) |x| - K\sigma|^{1+\alpha}}{2K\sigma}$$

New formulation of the chaotic intermittency

All points reinjected in the laminar region, $[0, c]$, are coming from the points close to x_r .

For $x_i \gtrsim x_r$, all points in the interval $(x_i, x_i + dx_i)$ are directly mapped into the interval $(F_2(x_i), F_2(x_i + dx_i)) \approx (x, x + F'_2(x_i)dx_i)$, where $x = F_2(x_i)$. Hence the probability to find a point in $(x_i, x_i + dx_i)$ is the same that to be reinjected into the interval $(x, x + F'_2(x_i)dx_i)$, where $F'_2(x)$ indicates the derivative of the function $F_2(x)$. Then, we have:

$$\lambda \rho(x_i) dx_i = \phi(x) F'_2(x_i) dx_i \quad (11)$$

The weight λ is introduced because the $\rho(x)$ density is normalized on the whole interval $[0, 1]$, whereas $\phi(x)$ is normalized only on the laminar interval. Finally, we can approximate $\phi(x)$ as follow

$$\phi(x) = \rho(x_i) \frac{\lambda}{\left. \frac{dF_2(\tau)}{d\tau} \right|_{\tau=x_i}},$$

where λ is a normalization constant such that $\int_0^c \phi(\tau) d\tau = 1$.

New formulation of the chaotic intermittency

- For $\alpha = 0$ we recover the most common approach with uniform RPD, i.e. $\phi(x) = \text{constant}$, widely considered in the literature.
- For $\alpha < 0$ the RPD increases without boundary for $x \rightarrow \hat{x}$.
- For $\alpha > 0$, the RPD decreases for $x \rightarrow \hat{x}$. In this last case, the two possibilities for the RPD, concave or convex, are separated by $\alpha = 1$

The RPD approaches to two limit cases:

$$\phi_0(x) = \lim_{\alpha \rightarrow -1} \phi(x) = \delta(x - \hat{x})$$

$$\phi_1(x) = \lim_{\alpha \rightarrow \infty} \phi(x) = \delta(x - c)$$

Classical formulation of the chaotic intermittency

The global map produces the reinjection of the trajectories in some point verifying $-c \leq x \leq c$. As the RPD is constant ($\phi(x) = k$) – uniform reinjection.

The average laminar length results:

Type-I:

$$\bar{l} = \frac{1}{\sqrt{a\varepsilon}} \arctan \left(c \sqrt{\frac{a}{\varepsilon}} \right)$$

Type-II:

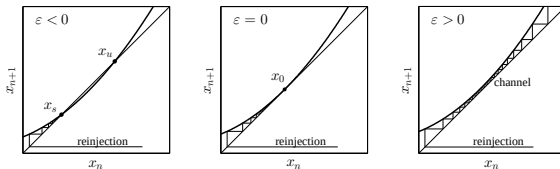
$$\bar{l} = \frac{1}{c\sqrt{a\varepsilon}} \arctan \left(c \frac{\sqrt{a}}{\sqrt{\varepsilon}} \right)$$

Type-III:

$$\bar{l} = \frac{1}{c\sqrt{a\varepsilon}} \arctan \left(c \frac{\sqrt{a}}{\sqrt{\varepsilon}} \right)$$

Type-I intermittency

The local map for type-I intermittency can be written as: $x_{n+1} = \varepsilon + x_n + a x_n^p$



For very small ε , we can use the continuous differential equation:

$$\frac{dx}{dl} = \varepsilon + a x^p,$$

from which we obtain $l = l(x, c)$ as a function of x

$$l(x, c) = \frac{c}{\varepsilon} {}_2F_1\left(\frac{1}{p}, 1; 1 + \frac{1}{p}; -\frac{ac}{\varepsilon}\right) - \frac{x}{\varepsilon} {}_2F_1\left(\frac{1}{p}, 1; 1 + \frac{1}{p}; -\frac{ax}{\varepsilon}\right)$$

in terms of the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$.

Type-I intermittency

In the case of $p = 2$, $l(x, c)$ can be given by

$$l(x, c) = \frac{1}{\sqrt{a\varepsilon}} \left[\tan^{-1} \left(\sqrt{\frac{a}{\varepsilon}} c \right) - \tan^{-1} \left(\sqrt{\frac{a}{\varepsilon}} x \right) \right]. \quad (12)$$

The probability density of the laminar lengths results:

$$\psi(l) = \phi(X(l, c)) \left| \frac{dX(l, c)}{dl} \right| = \phi(X(l, c)) |aX(l, c)^p + \varepsilon|$$

Where the explicit expression for $X(l, c)$ can be obtained only for a few cases. However, ψ can be plotted in all cases by using a parameterization procedure

$$(l(x, c), \psi'(x)) = (l(x, c), \phi(x) |\varepsilon + ax^p|).$$

where we have taken the coordinate of the reinjected points x as the free parameter.

Type-I intermittency

We describe the different shapes of $\psi(l)$ according with the values of α and $\hat{x}$.

Taking into account that the maximum length of $l = l_{max}$ is given for $x = \hat{x}$, we can determine the value of the function $\psi(l_{max})$:

$$\lim_{l \rightarrow l_{max}} \psi(l) = \lim_{x \rightarrow \hat{x}} \psi'(x) = \begin{cases} 0 & \text{if } \alpha > 0 \\ b(\varepsilon + a\hat{x}^p) & \text{if } \alpha = 0 \\ \infty & \text{if } \alpha < 0 \end{cases}$$

which depends on α .

The extreme points of the function $\psi(l)$ should be the root of the follow equation:

$$\frac{d\psi(l)}{dl} = \left((\varepsilon + ax^p) \frac{d\phi(x)}{dx} + ap\phi(x)x^{p-1} \right) \left| \frac{dX(l)}{dl} \right| = 0$$

Type-I intermittency

As $dX(I)/dI \neq 0$ for $\varepsilon \neq 0$, the expression between the square brackets must be zero for $x \in (\hat{x}, c)$ and $\varepsilon \approx 0$, hence the roots can be approximated as

$$x_{r1} \approx 0 \quad \text{and} \quad x_{r2} \approx \frac{p \hat{x}}{\alpha + p}.$$

This way, the extreme points for the density $\psi(I)$ can occur at two points, $L(x_{r1})$ and $L(x_{r2})$, provided that x_{r1} and x_{r2} lie in $(\hat{x}, c)$.

		$L(x_{r1})$	$L(x_{r2})$	$\lim_{I \rightarrow I_{max}} \psi(I)$	Subfigure
$\alpha > 0$	$\hat{x} > 0$	$\nexists$	$\nexists$	0	a
$\alpha > 0$	$\hat{x} < 0$	min	MAX	0	b
$\alpha < 0$	$\hat{x} > 0$	$\nexists$	min	∞	c
$\alpha < 0$	$\hat{x} < 0$	min	$\nexists$	∞	d
$\alpha = 0$	$\hat{x} < 0$	min	$\nexists$	$\frac{\varepsilon + a\hat{x}^p}{ \hat{x} - c }$	e
$\alpha = 0$	$\hat{x} > 0$	$\nexists$	$\nexists$	$\frac{\varepsilon + a\hat{x}^p}{ \hat{x} - c }$	f

Classification of the $\psi(I)$ local extreme types, minimum (min) or maximum (MAX), at $L(x_{r1})$ and $L(x_{r2})$, according to α and $\hat{x}$ values in the RPD. The limits $\lim_{I \rightarrow I_{max}} \psi(I)$, depending on α , are also given.

Type-I intermittency

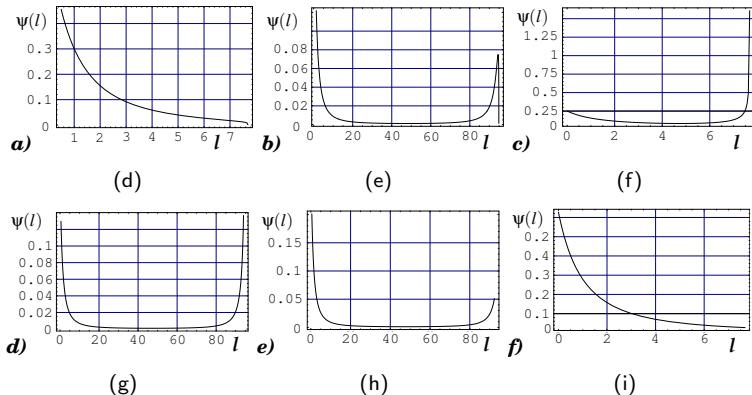


Figure : Different $\psi(l)$ profiles as function of $\hat{x}$ and α . The figures correspond to the parameters of the previous table. The numerical values selected to display the figures are: a) $\hat{x} = 0.1, \alpha = 0.2$, b) $\hat{x} = -0.4, \alpha = 0.3$, c) $\hat{x} = 0.1, \alpha = -0.6$, d) $\hat{x} = -0.4, \alpha = -0.3$, e) $\hat{x} = -0.3, \alpha = 0$ and f) $\hat{x} = 0.1, \alpha = 0$. In all cases $p = 2, a = 1, \varepsilon = 0.001$ and $c = 0.5$.

Noise effects on type-I intermittency

To illustrate the effect of noise in type-I intermittency we have chosen a model with direct reinjection, as happened in the type-II case:

$$F(x) = \begin{cases} F_1(x) = \varepsilon + x + ax^2 + \sigma\xi_n & \text{if } x < x_r, \\ F_2(x) = \hat{x} + \frac{1 - \hat{x}}{(1 - x_r)^\gamma} (x - x_r)^\gamma + \sigma\xi_n & \text{if } x > x_r \end{cases}$$

where x_r verifies: $\varepsilon + x_r + ax_r^2 = 1$.

The NRPD, $\Phi(x)$, is given by the following convolution integral:

$$\Phi(x) = \int \phi'(y) G(x - y, \sigma) dy,$$

$$\Phi(x) = \frac{b}{2\sigma(\alpha + 1)} \left\{ [x - (\hat{x} - \sigma)]^{\alpha+1} - \Theta[x - (\hat{x} + \sigma)] [x - (\hat{x} + \sigma)]^{\alpha+1} \right\}$$

where Θ is the Heaviside step function.

Type-II intermittency

We introduce a continuous differential equation to approximate the dynamics of the local map in the laminar region

$$x_{n+1} = (1 + \varepsilon)x_n + a x_n^p \quad \longrightarrow \quad \frac{dx}{dl} = \varepsilon x + a x^p$$

where l approximates the number of iterations in the laminar region, i.e. [the length of the laminar phase](#).

Note that l depends on the reinjection point x and it can be written as:

$$l(x, c) = \int_x^c \frac{d\tau}{a\tau^p + \varepsilon\tau}$$

After integration it yields

$$l(x, c) = \frac{1}{\varepsilon} \left[\ln \left(\frac{c}{x} \right) - \frac{1}{p-1} \ln \left(\frac{ac^{(p-1)} + \varepsilon}{ax^{(p-1)} + \varepsilon} \right) \right].$$

Length of laminar phase

The length of laminar phase of type-III is similar to the type-II intermittency. It is clear because we can write:

$$|x_{n+1}| = (1 + \varepsilon)|x_n| + a|x_n|^p$$

that gives the continuous differential equation

$$\frac{d|x|}{dl} = \varepsilon|x| + a|x|^3$$

where l indicates the number of iterations in the laminar region. Then, we can recover all the results on type-II intermittency, in particular the functions determining the exponent β .