# Spectral and pseudo-differential analysis of the boundary integral operators in 3D elastodynamics. Application to preconditioning

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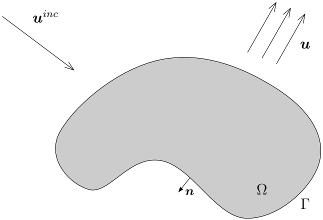
## Motivation

→ To construct a **fast solver** for exterior boundary value problems **at high frequency** $\omega$ :




$$\begin{cases} \mu\,\boldsymbol{\Delta}\,\boldsymbol{u} + (\lambda+\mu)\,\boldsymbol{\nabla}\,\mathrm{div}\,\boldsymbol{u} + \rho\omega^2\boldsymbol{u} & = \boldsymbol{0} \text{ in } \mathbb{R}^3\backslash\overline{\Omega} \\ \boldsymbol{u} + \boldsymbol{u}^{inc} & = \boldsymbol{0} \text{ on } \Gamma \\ + \text{ Kupradze radiation conditions} & \end{cases}$$

→ Reduction to boundary integral equations on $\Gamma$

→ **Drawbacks :**

1. The discretized matrix is not sparse and nonsymmetric
2. The required number of DOFs increases with the frequency $\Rightarrow$ large dimension $N$
   **Cost of iterative solver** : $\mathcal{O}(n_{iter}N^2)$

→ **Solutions :**

1. To speed up the matrix-vector product : Fast Multipole Method (Chaillat & al.)
2. To speed up the convergence of the iterative solver (GMRES)

## GMRES

Y. SAAD & M. H. SCHULTZ , *A generalized minimal residual algorithm for solving nonsymmetric linear systems*, SIAM (1986)

- **Algorithm** for solving $Ax = b$ :
  1. Choose an initial guess $x_0$ and compute the initial residual $r_0 = b - Ax_0$.
  2. While $||r_m|| < \text{tol}$ ,

     $$\text{Compute } x_m = \underset{x_m \in K_m(A, r_0)}{\text{argmin}} ||Ax_m - b||_2$$

     where $K_m(A, r_0)$ is the Krylov subspace

     $$K_m(A, r_0) = \text{span}\{r_0, Ar_0, \ldots, A^{m-1}r_0\} \ .$$

- **Error bounds** : if $A$ is diagonalisable and $A = VDV^{-1}$, then

  $$\frac{||r_m||}{||r_0||} \leqslant \text{cond}(V) \min_{p_m \in \mathcal{P}_m} \max_{\lambda \in \sigma(A)} |p_m(\lambda)| \ ,$$

  where $\mathcal{P}_m$ is the set of all polynomials of degree $m$ or less over $\mathbb{C}$ satisfying $p(0) = 1$.

- **Fast convergence** provided
  1. $\text{cond}(V) \approx 1$ or $A \approx \text{normal matrix}$
  2. eigenvalue clustering about some nonzero point $c \in \mathbb{C}$.

# Outline

D. KUPRADZE & , *Three-dimensional problems of the mathematical theory of elasticity ...*

- The time-harmonic Navier equation : $\mu\,\boldsymbol{\Delta}\,\boldsymbol{u} + (\lambda+\mu)\,\boldsymbol{\nabla}\,\mathrm{div}\,\boldsymbol{u} + \rho\omega^2\boldsymbol{u} = 0$.
- $G(\kappa, x-y) = \frac{e^{i\kappa|x-y|}}{4\pi|x-y|}$ , $\kappa_p = \omega\sqrt{\mu\rho^{-1}}$ and $\kappa_s = \omega\sqrt{(\lambda+2\mu)\rho^{-1}}$. The Green function is

$$\Phi(\omega, x-y) = \frac{1}{\rho\omega^2}\left(\mathbf{curl}\,\mathbf{curl}_x\left\{G(\kappa_s, x-y)\,\mathrm{I}_3\right\} - \boldsymbol{\nabla}_x\,\mathrm{div}_x\,\left\{G(\kappa_p, x-y)\mathrm{I}_3\right\}\right).$$

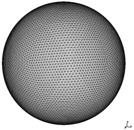
- Traction trace : $\boldsymbol{T} = 2\mu\frac{\partial}{\partial\boldsymbol{n}} + \lambda\boldsymbol{n}\,\mathrm{div} + \mu\,\boldsymbol{n}\times\mathbf{curl}$.
- The elastodynamic potential operators are, for $x \in \mathbb{R}^3\backslash\overline{\Omega}$,

$$\mathscr{S}\boldsymbol{\varphi}(x) = \int_\Gamma \Phi(\omega, x-y)\boldsymbol{\varphi}(y)d\sigma(y)\ , \quad \mathscr{D}\boldsymbol{\psi}(x) = \int_\Gamma {}^{\mathsf{T}}[\boldsymbol{T}_y\Phi(\omega, x-y)]\boldsymbol{\psi}(y)d\sigma(y)$$

- Integral representation of the solution : $\boxed{\boldsymbol{u} = \mathscr{D}\boldsymbol{\psi} + i\eta\mathscr{S}\boldsymbol{\psi}}$ where $\eta \neq 0$ to ensure uniqueness.
- Solving the Direct Problem $\Leftrightarrow$ solving a Boundary Integral Equation

$$\boldsymbol{u} + \boldsymbol{u}^{inc} = 0 \quad \Rightarrow \quad \frac{1}{2}\boldsymbol{\psi}(x) + D\boldsymbol{\psi}(x) + i\eta S\boldsymbol{\psi}(x) = -\boldsymbol{u}^{inc}(x)\ ,\ x \in \Gamma$$

$$S_\kappa\boldsymbol{\varphi}(x) = \lim_{s\to 0}(\mathscr{S}\boldsymbol{\varphi})(x + s\boldsymbol{n}(x)) \text{ and } \frac{1}{2}\boldsymbol{\psi}(x) + D\boldsymbol{\psi}(x) = \lim_{s\to 0}(\mathscr{D}\boldsymbol{\psi})(x + s\boldsymbol{n}(x))$$



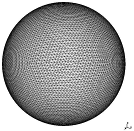
- Diffraction of an incident plane P-wave by the unit sphere

$$\boldsymbol{u}^{inc}(x) = \boldsymbol{p}e^{i\kappa_p x\cdot \boldsymbol{d}}, \quad \boldsymbol{p} = \boldsymbol{d} = {}^{\mathsf{T}}(0,0,1)$$

- Simulations performed with a FM-BEM code [Chaillat-Bonnet-Semblat, 08].
- GMRES solver with no restart and tolerance $\varepsilon = 10^{-3}$.
- $\rho = 1$, $\mu = 1$ et $\lambda = 0.25$ and $\kappa_s = 1.5\kappa_p$
- Wavenumber $\lambda_s = \frac{2\pi}{\kappa_s}$.
- Number of DOFs per wavelenght : $n_{\lambda_s} = 10$
- CFIE parameter : $\eta = 1$
- Number of GMRES iteration :

| #DOFs | $\omega$ | # iter |
|---|---|---|
| 1 926 | 4 | 18 |
| 7 686 | 8.25 | 27 |
| 30 726 | 16.5 | 51 |
| 122 886 | 33 | 180 |
| 490 629 | 66.5 | $>$ 500 |

- **GMRES convergence depends on both a frequency increase and a mesh refinement**



- eigenvalue repartition of the CFIE :

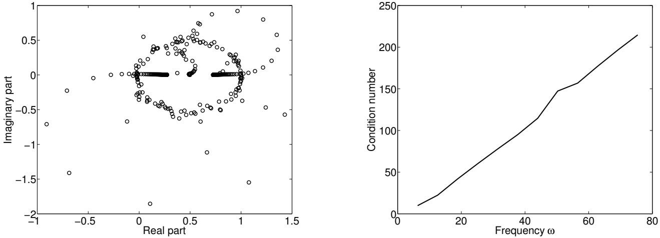



FIGURE: (**Left**) : Distribution of the eigenvalues of the standard CFIE ($\eta = 1$, $\kappa_s = 16\pi$ and $n_{\lambda_s} = 10$). (**Right**) : Condition number with respect to $\omega$ ($n_{\lambda_s} = 10$).

- **How to speed up the convergence of the GMRES ?** The discretized matrix must have :
  1. eigenvalue clustering about $c = 1$ ,
  2. a condition number close to 1.

X. ANTOINE & M. DARBAS , *Generalized combined field integral equations for the iterative solution of the three-dimensional Helmholtz equation.*

- **We consider a new layer ansatz** :

$$\boxed{\boldsymbol{u}(x) = \mathscr{D}\boldsymbol{\varphi}(x) - \mathscr{S}\boldsymbol{\Lambda}\boldsymbol{\varphi}(x)}\,, \qquad x \in \mathbb{R}^3\backslash\overline{\Omega}$$

$$\Rightarrow \quad -\boldsymbol{u}^{inc}_{|\Gamma}(x) = (\tfrac{1}{2}\mathrm{I} + D)\boldsymbol{\varphi}(x) - S\boldsymbol{\Lambda}\boldsymbol{\varphi}(x)\,, \qquad x \in \Gamma$$

where the operator $\boldsymbol{\Lambda}$ is chosen sucht that $\boxed{(\tfrac{1}{2}\mathrm{I} + D) - S\boldsymbol{\Lambda} \approx \mathrm{I}}$.

- **Extreme case** :
  1. $\mathrm{I} = (\tfrac{1}{2}\mathrm{I} + D) - S\boldsymbol{\Lambda} \Rightarrow \boldsymbol{\Lambda} := -S^{-1}(\tfrac{1}{2}\mathrm{I} - D)$ (*if $\omega$ is not an eigenfrequency*)
  2. The **Dirichlet-to-Neumann map** is defined by :

$$DtN : \boldsymbol{u}_{|\Gamma} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma) \mapsto (\boldsymbol{T}\boldsymbol{u})_{|\Gamma} := DtN\boldsymbol{u}_{|\Gamma} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma).$$

Then, the Somigliana integral representation formula leads to

$$\boldsymbol{u}(x) = \mathscr{D}\boldsymbol{u}_{|\Gamma} - \mathscr{S}(\boldsymbol{T}\boldsymbol{u})_{|\Gamma} \quad \Rightarrow \quad \boldsymbol{u}_{|\Gamma} = -\boldsymbol{u}^{inc}_{|\Gamma} = (\tfrac{1}{2}\mathrm{I} + D)\boldsymbol{u}_{|\Gamma} - SDtN\boldsymbol{u}_{|\Gamma}$$

$$\Rightarrow \quad \boldsymbol{\Lambda} = DtN \text{ and } \boldsymbol{\varphi} = \boldsymbol{u}_{|\Gamma}$$

→ The exact DtN operator $\boldsymbol{\Lambda}^{\mathrm{ex}}$ is a good preconditioner for the boundary integral equation

- **Objective** : we use, instead, the (pseudo-differential) principal part of the $DtN$ map, that is easy to implement. We get $\boxed{(\tfrac{1}{2}\mathrm{I} + D) - S\boldsymbol{\Lambda} = \mathrm{I} + \textbf{compact}}$

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D. KUPRADZE , *three-dimensional problems of the mathematical theory of elasticity ...*

- The elastodynamic boundary integral operators are :

$$S\varphi(x) = \int_\Gamma \Phi(\omega, x-y)\varphi(y)d\sigma(y)\ , \quad D\psi(x) = \int_\Gamma {}^{\mathsf{T}}[T_y\Phi(\omega, x-y)]\psi(y)d\sigma(y)$$

$$D'\varphi(x) = \int_\Gamma [T_x\Phi(\omega, x-y)]\varphi(y)d\sigma(y)\ , \quad N\psi(x) = \int_\Gamma T_x{}^{\mathsf{T}}[T_y\Phi(\omega, x-y)]\psi(y)d\sigma(y)$$

- Mathematical properties :
  1. $S : H^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$ is compact.
     *The eigenvalues of $S$ form an increasing sequence that tends to zero.*
  2. $D : H^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$ and $D' : H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ are not compact.
     *The eigenvalues of $D$ and $D'$ are bounded.*
  3. $N : H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$ is unbounded from $H^{\frac{1}{2}}(\Gamma)$ to itself .
     *The eigenvalues of $N$ form an increasing sequence that tends to infinity.*
- Calderón relations :

$$SN = -\frac{1}{4}\mathrm{I} + D^2 \qquad NS = -\frac{1}{4}\mathrm{I} + D'^2$$

$$SD' = DS \qquad ND = D'N$$

- Various representations of the exact DtN operator : *if $\omega$ is not an eigenfrequency, then*

$$DtN := -S^{-1}(\frac{1}{2}\mathrm{I} - D) \qquad \text{or} \qquad DtN := N(\frac{1}{2}\mathrm{I} + D)^{-1}$$

M. S. Agranovich & B. A. Amosov & M. Levitin, *Spectral problems for the Lamé system with spectral parameter in boundary conditions on smooth or nonsmooth boundary*

- $L(\partial_x) = \mu\,\boldsymbol{\Delta} + (\lambda+\mu)\,\boldsymbol{\nabla}\,\mathrm{div} + \rho\omega^2\mathrm{I}$
- $\Delta = \boldsymbol{\nabla}\,\mathrm{div} - \mathbf{curl}\,\mathbf{curl} \Rightarrow L(\partial_x) = (\lambda+2\mu)\,\boldsymbol{\nabla}\,\mathrm{div} - \mu\,\mathbf{curl}\,\mathbf{curl} + \rho\omega^2\mathrm{I}$
- It has the symbol $\ell(\boldsymbol{\xi}) = -(\lambda+2\mu)\boldsymbol{\xi}^{\mathsf{T}}\boldsymbol{\xi} - \mu(|\boldsymbol{\xi}|^2 - \boldsymbol{\xi}^{\mathsf{T}}\boldsymbol{\xi}) + \rho\omega^2\mathrm{I}$ , that can be written :

$$\ell(\boldsymbol{\xi}) = (\lambda+2\mu)\left(\kappa_p^2 - |\boldsymbol{\xi}|^2\right)\frac{1}{|\boldsymbol{\xi}|^2}\boldsymbol{\xi}^{\mathsf{T}}\boldsymbol{\xi} + \mu\left(\kappa_s^2 - |\boldsymbol{\xi}|^2\right)\left(\mathrm{I} - \frac{1}{|\boldsymbol{\xi}|^2}\boldsymbol{\xi}^{\mathsf{T}}\boldsymbol{\xi}\right)$$

- Th volume potential with kernel $\Phi(\omega, x-y)$ has the symbol

$$\ell^{-1}(\boldsymbol{\xi}) = (\lambda+2\mu)^{-1}\frac{-1}{|\boldsymbol{\xi}|^2-\kappa_p^2}\frac{1}{|\boldsymbol{\xi}|^2}\boldsymbol{\xi}^{\mathsf{T}}\boldsymbol{\xi} + \mu^{-1}\frac{-1}{|\boldsymbol{\xi}|^2-\kappa_s^2}\left(\mathrm{I} - \frac{1}{|\boldsymbol{\xi}|^2}\boldsymbol{\xi}^{\mathsf{T}}\boldsymbol{\xi}\right)$$

- The boundary integral operator with kernel $\Phi(\omega, x-y)$ has the principal symbol

$$\sigma(S) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\ell^{-1}(\boldsymbol{\xi})d\xi_3 \qquad \text{where } \boldsymbol{\xi} = (\xi_1,\xi_2,\xi_3) = (\boldsymbol{\xi}_{||},\xi_3)$$

$$= \frac{1}{2\rho\omega^2}\left(\frac{-1}{\sqrt{|\boldsymbol{\xi}_{||}|^2-\kappa_p^2}}\begin{pmatrix}\boldsymbol{\xi}_{||}{}^{\mathsf{T}}\boldsymbol{\xi}_{||} & 0\\ 0 & |\boldsymbol{\xi}_{||}|^2-\kappa_p^2\end{pmatrix} + \frac{-1}{\sqrt{|\boldsymbol{\xi}_{||}|^2-\kappa_s^2}}\begin{pmatrix}\kappa_s^2\mathrm{I}_{||} - \boldsymbol{\xi}_{||}{}^{\mathsf{T}}\boldsymbol{\xi}_{||} & 0\\ 0 & -|\boldsymbol{\xi}_{||}|^2\end{pmatrix}\right)$$

- We deduce the principal part of $S$ in terms of differential operators

$$P_{-1}(S) = \frac{i}{2\rho\omega^2}\Big[-\boldsymbol{\nabla}_\Gamma\,(\Delta_\Gamma + \kappa_p^2\mathrm{I})^{-\frac{1}{2}}\,\mathrm{div}_\Gamma\,\mathrm{I}_t + \boldsymbol{n}(\Delta_\Gamma + \kappa_p^2\mathrm{I})^{\frac{1}{2}}(\boldsymbol{n}\cdot\mathrm{I}_{\boldsymbol{n}}) + (\boldsymbol{\Delta}_\Gamma + \kappa_s^2\mathrm{I})^{-\frac{1}{2}}\left(\kappa_s^2\mathrm{I}_t + \boldsymbol{\nabla}_\Gamma\,\mathrm{div}_\Gamma\,\mathrm{I}_t\right) - \boldsymbol{n}(\Delta_\Gamma + \kappa_s^2\mathrm{I})^{-\frac{1}{2}}\Delta_\Gamma\,(\boldsymbol{n}\cdot\mathrm{I}_{\boldsymbol{n}})\Big]$$

G. C. Hsiao & W. L. Wendland , *Boundary integral equations*

- Tangential Günter derivative : $\mathcal{M} = \dfrac{\partial}{\partial \boldsymbol{n}} - \boldsymbol{n}\,\mathrm{div} + \boldsymbol{n} \times \mathbf{curl} = [\boldsymbol{\nabla}_\Gamma \cdot\,]\boldsymbol{n} - \boldsymbol{n}\,\mathrm{div}_\Gamma \cdot$
- Traction trace : $\boldsymbol{T} = 2\mu \dfrac{\partial}{\partial \boldsymbol{n}} + \lambda \boldsymbol{n}\,\mathrm{div} + \mu\, \boldsymbol{n} \times \mathbf{curl} = 2\mu\mathcal{M} + (\lambda + 2\mu)\boldsymbol{n}\,\mathrm{div} - \mu\, \boldsymbol{n} \times \mathbf{curl}$
- Integral representation of the double layer boundary integral operator :

$$\begin{aligned}(D\boldsymbol{\psi})(x) \quad &= \quad 2\mu S\mathcal{M}\boldsymbol{\psi}(x) - \int_\Gamma {}^{\mathsf{T}}[\boldsymbol{n}(y) \times \mathbf{curl}_y\{G(\kappa_s, x-y)\mathrm{I}_3\}]\boldsymbol{\psi}(y)ds(y) \\ &\quad - \int_\Gamma \boldsymbol{\nabla}_x\, G(\kappa_p, x-y)\big(\boldsymbol{n}(y)\cdot\boldsymbol{\psi}(y)\big)\, ds(y),\end{aligned}$$

- Well-known expansions :
  1. $\boldsymbol{\nabla}\, v = \boldsymbol{\nabla}_\Gamma\, v + \boldsymbol{n}\dfrac{\partial v}{\partial \boldsymbol{n}},$
  2. $\boldsymbol{n} \times \mathbf{curl}\, \boldsymbol{u} = \boldsymbol{\nabla}_\Gamma(\boldsymbol{u}\cdot\boldsymbol{n}) + \boldsymbol{n} \times (\mathcal{R} - 2\mathcal{H}\,\mathrm{I}_3)(\boldsymbol{u}\times\boldsymbol{n}) - \boldsymbol{n} \times \left(\dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} \times \boldsymbol{n}\right)$
- We deduce the principal part of $D$ in terms of differential operators

$$P_0(D) = 2\mu P_{-1}(S)\mathcal{M} + \frac{i}{2}\Big(\boldsymbol{n}\big(\Delta_\Gamma + \kappa_s^2\mathrm{I}\big)^{-\frac{1}{2}}\,\mathrm{div}_\Gamma\,\mathrm{I}_t - \boldsymbol{\nabla}_\Gamma\,\big(\Delta_\Gamma + \kappa_p^2\mathrm{I}\big)^{-\frac{1}{2}}\,\boldsymbol{n}\cdot\mathrm{I}_{\boldsymbol{n}}\Big)$$

- $\boldsymbol{\Lambda} = -P_{-1}(S)^{-1}(\frac{1}{2}\mathrm{I} - P_0(D))$
- **Remark :** $P_0(D - 2\mu S\mathcal{M}) = \dfrac{i}{2}\Big(\boldsymbol{n}\big(\Delta_\Gamma + \kappa_s^2\mathrm{I}\big)^{-\frac{1}{2}}\,\mathrm{div}_\Gamma\,\mathrm{I}_t - \boldsymbol{\nabla}_\Gamma\,\big(\Delta_\Gamma + \kappa_p^2\mathrm{I}\big)^{-\frac{1}{2}}\,\boldsymbol{n}\cdot\mathrm{I}_{\boldsymbol{n}}\Big)$

M. DARBAS & F. LE LOUËR , *Well-conditioned boundary integral formulations for high-frequency elastic scattering problems in three dimensions*

- **We consider a new layer ansatz** :

$$\boxed{\boldsymbol{u}(x) = (\mathscr{D} - 2\mu\mathscr{S}\mathcal{M})\boldsymbol{\varphi}(x) - \mathscr{S}(\boldsymbol{\Lambda} - 2\mu\mathcal{M})\boldsymbol{\varphi}(x)}\,, \qquad x \in \mathbb{R}^3\backslash\overline{\Omega}$$

$$\Rightarrow \quad -\boldsymbol{u}^{inc}_{|\Gamma}(x) = (\tfrac{1}{2}\mathrm{I} + (D - 2\mu S\mathcal{M}))\boldsymbol{\varphi}(x) - S(\boldsymbol{\Lambda} - 2\mu\mathcal{M})\boldsymbol{\varphi}(x)\,, \qquad x \in \Gamma$$

where the operator $\boldsymbol{\Lambda}$ is chosen sucht that $\boxed{\left(\tfrac{1}{2}\mathrm{I} + (D - 2\mu S\mathcal{M})\right) - S(\boldsymbol{\Lambda} - 2\mu\mathcal{M}) \approx \mathrm{I}}$.

- **Extreme case :** $\boldsymbol{\Lambda} \approx DtN = -S^{-1}\left(\tfrac{1}{2}\mathrm{I} - (D - 2\mu S\mathcal{M})\right) + 2\mu\mathcal{M}$

- **New results for the elastodynamic potential theory** :
  1. Neumann-type trace : $\boldsymbol{T}_{2\mu} = \boldsymbol{T} - 2\mu\mathcal{M} = (\lambda + 2\mu)\boldsymbol{n}\,\mathrm{div} - \mu\boldsymbol{n} \times \mathbf{curl}$.
  2. New boundary integral operators : $S$, $\boxed{D_{2\mu} = D - 2\mu S\mathcal{M}}$, $D'_{2\mu} = D' - 2\mu\mathcal{M}S$ and $\boxed{N_{2\mu} = N - 2\mu(D'\mathcal{M} + \mathcal{M}D) + (2\mu)^2\mathcal{M}S\mathcal{M}}$
  3. Calderon relations : $SN_{2\mu} = -\tfrac{1}{4}\mathrm{I} + D^2_{2\mu} \Rightarrow \boxed{DtN := N_{2\mu}\left(\tfrac{1}{2}\mathrm{I} + D_{2\mu}\right)^{-1} + 2\mu\mathcal{M}}$
  4. The associated **Dirichlet-to-Neumann map** is defined by :

$$\boldsymbol{u}_{|\Gamma} = (\mathbf{curl}\,\boldsymbol{\psi}_s + \nabla\psi_p)_{|\Gamma} \mapsto (\boldsymbol{T}_{2\mu}\boldsymbol{u})_{|\Gamma} := -\rho\omega^2(\boldsymbol{n}\psi_{p_{|\Gamma}} + \boldsymbol{n} \times \boldsymbol{\psi}_s).$$

$\rightarrow$ It contains the acoustic and electromagnetic Neumann-to-Dirichlet maps !

- **Integral representation** : $\boldsymbol{T}_{2\mu} = \boldsymbol{T} - 2\mu\mathcal{M} = (\lambda + 2\mu)\boldsymbol{n}\,\mathrm{div} - \mu\boldsymbol{n} \times \mathbf{curl}$

$$\begin{aligned} N_{2\mu}\boldsymbol{\varphi}(x) &= \int_\Gamma \boldsymbol{T}_{2\mu,x}{}^{\top}[\boldsymbol{T}_{2\mu,y}\Phi(\omega, x-y)]\boldsymbol{\psi}(y)d\sigma(y) \\ &= \rho\omega^2 \int_\Gamma \boldsymbol{n}(x)G(\kappa_p, x-y)\big(\boldsymbol{n}(y)\cdot\boldsymbol{\varphi}(y)\big)ds(y) \\ &\quad + \rho\omega^2\,\boldsymbol{n}(x)\times \int_\Gamma G(\kappa_s, x-y)(\boldsymbol{\varphi}(y)\times\boldsymbol{n}(y))ds(y) \\ &\quad - \mu\,\mathbf{curl}_\Gamma \int_\Gamma G(\kappa_s, x-y)\,\mathrm{curl}_\Gamma\boldsymbol{\varphi}(y)ds(y), \qquad x\in\Gamma, \end{aligned}$$

  where we have $\mathbf{curl}_\Gamma = -\boldsymbol{n}\times\boldsymbol{\nabla}_\Gamma$ and $\mathrm{curl}_\Gamma = \mathrm{div}_\Gamma(\cdot\times\boldsymbol{n})$.

- **Principal part of $N_{2\mu}$**

$$P_1(N_{2\mu}) = \frac{i\rho\omega^2}{2}\left[\boldsymbol{n}\big(\Delta_\Gamma + \kappa_p^2\mathrm{I}\big)^{-\frac{1}{2}}\boldsymbol{n}\cdot\mathrm{I}_{\boldsymbol{n}} + \big(\boldsymbol{\Delta}_\Gamma + \kappa_s^2\mathrm{I}\big)^{-\frac{1}{2}}\left(\mathrm{I}_{\boldsymbol{t}} - \frac{1}{\kappa_s^2}\mathbf{curl}_\Gamma\,\mathrm{curl}_\Gamma\right)\right].$$

- **The preconditionner is**

$$\boldsymbol{\Lambda} = P_1(N_{2\mu})\big(\tfrac{1}{2}\mathrm{I} + P_0(D_{2\mu})\big)^{-1} + 2\mu\mathcal{M}$$

  with

$$P_0(D_{2\mu}) = \frac{i}{2}\Big(-\boldsymbol{\nabla}_\Gamma\,(\Delta_\Gamma + \kappa_p^2\mathrm{I})^{-\frac{1}{2}}\,\boldsymbol{n}\cdot\mathrm{I}_{\boldsymbol{n}} + \boldsymbol{n}(\Delta_\Gamma + \kappa_s^2\mathrm{I})^{-\frac{1}{2}}\,\mathrm{div}_\Gamma\,\mathrm{I}_{\boldsymbol{t}}\Big)$$

- **Drawback** : $\big(\frac{1}{2}\mathrm{I} + P_0(D_{2\mu})\big)$ has a very bad eigenvalue clustering.

M. Darbas & F. Le Louër , *Well-conditioned boundary integral formulations for high-frequency elastic scattering problems in three dimensions*

- Choose $\alpha \neq 2\mu$.
- **We consider a new layer ansatz** :

$$\boxed{\boldsymbol{u}(x) = (\mathscr{D} - \alpha \mathscr{S}\mathcal{M})\boldsymbol{\varphi}(x) - \mathscr{S}(\boldsymbol{\Lambda} - \alpha\mathcal{M})\boldsymbol{\varphi}(x)}\,, \qquad x \in \mathbb{R}^3\backslash\overline{\Omega}$$

$$\Rightarrow \quad -\boldsymbol{u}^{inc}_{|\Gamma}(x) = (\tfrac{1}{2}\mathrm{I} + (D - \alpha S\mathcal{M}))\boldsymbol{\varphi}(x) - S(\boldsymbol{\Lambda} - \alpha\mathcal{M})\boldsymbol{\varphi}(x)\,, \qquad x \in \Gamma$$

where the operator $\boldsymbol{\Lambda}$ is chosen sucht that $\boxed{\big(\tfrac{1}{2}\mathrm{I} + (D - \alpha S\mathcal{M})\big) - S(\boldsymbol{\Lambda} - \alpha\mathcal{M}) \approx \mathrm{I}}$.

- **Extreme case :** $\boldsymbol{\Lambda} \approx DtN = -S^{-1}\big(\tfrac{1}{2}\mathrm{I} - (D - \alpha S\mathcal{M})\big) + \alpha\mathcal{M}$
- **New results for the elastodynamic potential theory** :
  1. Neumann-type trace : $\boldsymbol{T}_\alpha = \boldsymbol{T} - \alpha\mathcal{M} = (2\mu - \alpha)\mathcal{M} + (\lambda + 2\mu)\boldsymbol{n}\,\mathrm{div} - \mu\boldsymbol{n} \times \mathbf{curl}$.
  2. New boundary integral operators : $S$, $\boxed{D_\alpha = D_{2\mu} - (\alpha - 2\mu)S\mathcal{M}}$, $D'_\alpha = D'_{2\mu} - (\alpha - 2\mu)\mathcal{M}S$

     and $\boxed{N_\alpha = N_{2\mu} - (\alpha - 2\mu)(D'_{2\mu}\mathcal{M} + \mathcal{M}D_{2\mu}) + (\alpha - 2\mu)^2\mathcal{M}S\mathcal{M}}$
  3. Calderon relations : $SN_\alpha = -\tfrac{1}{4}\mathrm{I} + D^2_\alpha \Rightarrow \boxed{DtN := N_\alpha(\tfrac{1}{2}\mathrm{I} + D_\alpha)^{-1} + \alpha\mathcal{M}}$
  4. The preconditionner is

$$\boldsymbol{\Lambda} = P_1(N_\alpha)\big(\tfrac{1}{2}\mathrm{I} + P_0(D_\alpha)\big)^{-1} + \alpha\mathcal{M}$$

- **Choose the best value** $\alpha$ **:** when $\alpha = \frac{2\mu^2}{3\lambda+2\mu}$, then $D_\alpha$ is compact !

  P. HÄHNER & G. C. HSIAO , *Uniqueness theorems in inverse obstacle scattering of elastic waves*

  → We use the GMRES solver to invert $(\frac{1}{2}\mathrm{I} + P_0^\varepsilon(D_\alpha))$.

- **Regularization procedure :** we set $\kappa_{s,\varepsilon} = \kappa_s + i\varepsilon_s$ and $\kappa_{p,\varepsilon} = \kappa_s + i\varepsilon_p$.
  We obtain a better eigenvalue clustering in the grazing zone $\beta_n \approx \kappa_p$ or $\beta_n \approx \kappa_s$

- **Optimal values of** $\varepsilon_s$ **and** $\varepsilon_p$ : $\varepsilon_\gamma = 0.39\kappa_\gamma^{1/3}(\mathcal{H}^2)^{1/3}$ for $\gamma = s, p$

- **Final expression :** $\boldsymbol{\Lambda} = P_1^\varepsilon(N_\alpha)\big(\frac{1}{2}\mathrm{I} + P_0^\varepsilon(D_\alpha)\big)^{-1} + \alpha\mathcal{M}$

$$
\begin{aligned}
P_0^\varepsilon(D_\alpha) &= P_0^\varepsilon(D_{2\mu}) - (\alpha - 2\mu)P_{-1}^\varepsilon(S)\mathcal{M}_P \\
P_1^\varepsilon(N_\alpha) &= P_1^\varepsilon(N_{2\mu}) - (\alpha - 2\mu)(P_0^\varepsilon(D'_{2\mu})\mathcal{M}_P + \mathcal{M}_P P_0^\varepsilon(D_{2\mu})) + (\alpha - 2\mu)^2\mathcal{M}_P P_{-1}^\varepsilon(S)\mathcal{M}_P
\end{aligned}
$$

with

$$
\begin{aligned}
P_{-1}^\varepsilon(S) &= \frac{i}{2\rho\omega^2}\Big[ -\boldsymbol{\nabla}_\Gamma\,(\Delta_\Gamma + \kappa_{p,\varepsilon}^2\mathrm{I})^{-\frac{1}{2}}\,\mathrm{div}_\Gamma\,\mathrm{I}_t + \boldsymbol{n}(\Delta_\Gamma + \kappa_{p,\varepsilon}^2\mathrm{I})^{\frac{1}{2}}\,(\boldsymbol{n}\cdot\mathrm{I}_{\boldsymbol{n}}) \\
&\qquad + (\boldsymbol{\Delta}_\Gamma + \kappa_{s,\varepsilon}^2\mathrm{I})^{-\frac{1}{2}}\left(\kappa_{s,\varepsilon}^2\mathrm{I}_t + \boldsymbol{\nabla}_\Gamma\,\mathrm{div}_\Gamma\,\mathrm{I}_t\right) - \boldsymbol{n}(\Delta_\Gamma + \kappa_{s,\varepsilon}^2\mathrm{I})^{-\frac{1}{2}}\,\Delta_\Gamma\,(\boldsymbol{n}\cdot\mathrm{I}_{\boldsymbol{n}})\Big] \\
\mathcal{M}_P &= \boldsymbol{\nabla}_\Gamma(\boldsymbol{n}\,\cdot\,) - \boldsymbol{n}\,\mathrm{div}_\Gamma\,\mathrm{I}_t \\
P_0^\varepsilon(D_{2\mu}) &= \frac{i}{2}\Big( -\boldsymbol{\nabla}_\Gamma\,(\Delta_\Gamma + \kappa_{p,\varepsilon}^2\mathrm{I})^{-\frac{1}{2}}\,\boldsymbol{n}\cdot\mathrm{I}_{\boldsymbol{n}} + \boldsymbol{n}(\Delta_\Gamma + \kappa_{s,\varepsilon}^2\mathrm{I})^{-\frac{1}{2}}\,\mathrm{div}_\Gamma\,\mathrm{I}_t\Big) \\
P_1^\varepsilon(N_{2\mu}) &= \frac{i\rho\omega^2}{2}\Big[\boldsymbol{n}(\Delta_\Gamma + \kappa_{p,\varepsilon}^2\mathrm{I})^{-\frac{1}{2}}\,\boldsymbol{n}\cdot\mathrm{I}_{\boldsymbol{n}} + (\boldsymbol{\Delta}_\Gamma + \kappa_{s,\varepsilon}^2\mathrm{I})^{-\frac{1}{2}}\Big(\mathrm{I}_t - \frac{1}{\kappa_{s,\varepsilon}^2}\,\mathbf{curl}_\Gamma\,\mathrm{curl}_\Gamma\Big)\Big]
\end{aligned}
$$

- **Complete expression :** $\boldsymbol{\Lambda}_{HO}^{(2)} = P_1^{\varepsilon}(N_\alpha)\left(\frac{1}{2}\mathrm{I} + P_0^{\varepsilon}(D_\alpha)\right)^{-1} + \alpha\mathcal{M}$

- **We omit the grazing zone :** $\boldsymbol{\Lambda}_{HO}^{(1)} = P_1^{\varepsilon}(N_\alpha) + \alpha\mathcal{M}$

- **We omit the grazing zone and the elliptic zone :**

$$\boldsymbol{\Lambda}_{GG} := i((\lambda + 2\mu)\kappa_p \mathrm{I}_{\boldsymbol{n}} + \mu\kappa_s \mathrm{I}_{\boldsymbol{t}}$$

G. K. GÄCHTER & M. J. GROTE , *Dirichlet-to-Neumann map for three-dimensional elastic waves*

→ $\boldsymbol{\Lambda}_{GG}$ is contained in $P_1(N_{2\mu})$ :

$$P_1(N_{2\mu}) = \frac{i\rho\omega^2}{2}\left[\boldsymbol{n}\left(\Delta_\Gamma + \kappa_p^2\mathrm{I}\right)^{-\frac{1}{2}}\boldsymbol{n}\cdot\mathrm{I}_{\boldsymbol{n}} + \left(\boldsymbol{\Delta}_\Gamma + \kappa_s^2\mathrm{I}\right)^{-\frac{1}{2}}\left(\mathrm{I}_{\boldsymbol{t}} - \frac{1}{\kappa_s^2}\,\mathbf{curl}_\Gamma\,\mathrm{curl}_\Gamma\right)\right]$$

- **Standard CFIE** : $\boldsymbol{\Lambda} = i\eta\mathrm{I},\ \eta \in \mathbb{R},\ \eta \neq 0.$

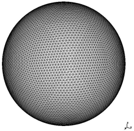
- Diffraction of an incident plane P-wave by the unit sphere

$$\boldsymbol{u}^{inc}(x) = \boldsymbol{p}e^{i\kappa_p x\cdot\boldsymbol{d}}, \quad \boldsymbol{p} = \boldsymbol{d} = {}^{\mathsf{T}}(0,0,1)$$

- Diffraction of an incident plane S-wave by the unit sphere

$$\boldsymbol{u}^{inc}(x) = \boldsymbol{p}e^{i\kappa_p x\cdot\boldsymbol{d}}, \quad \boldsymbol{d} = {}^{\mathsf{T}}(0,0,1) \quad \boldsymbol{p} = {}^{\mathsf{T}}(1,0,0)$$

- Simulations performed with a FM-BEM code [Chaillat-Bonnet-Semblat, 08].
- GMRES solver with no restart and tolerance $\varepsilon = 10^{-3}$.
- $\rho = 1$, $\mu = 1$ et $\lambda = 0.25$ and $\kappa_s = 1.5\kappa_p$
- wavenumber $\lambda_s = \frac{2\pi}{\kappa_s}$.
- fixed density of points per wavelength $n_{\lambda_s} = 10$

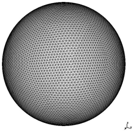


| #DOFs | $\omega$ | # iter CFIE | # iter GG P-CFIE | # iter HO(1) P-CFIE | # iter HO(2) P-CFIE |
|---|---|---|---|---|---|
| 1 926 | 4 | 18 | 8 | 7 | 5 (11) |
| 7 686 | 8.25 | 27 | 8 | 6 | 4 (11) |
| 30 726 | 16.5 | 51 | 9 | 6 | 3 (13) |
| 122 886 | 33 | 180 | 9 | 6 | 3 (13) |
| 490 629 | 66.5 | $>$ 500 | 9 | 6 | 3 (14) |

TABLE: Diffraction of P-waves by the unit sphere. Number of GMRES iterations for a fixed density of points per wavelength.

| #DOFs | $\omega$ | # iter CFIE | # iter GG P-CFIE | # iter HO(1) P-CFIE | # iter HO(2) P-CFIE |
|---|---|---|---|---|---|
| 1 926 | 4 | 18 | 9 | 7 | 6 (10) |
| 7 686 | 8.25 | 26 | 10 | 7 | 4 (11) |
| 30 726 | 16.5 | 75 | 11 | 7 | 4 (14) |
| 122 886 | 33 | 199 | 14 | 8 | 4 (15) |
| 490 629 | 66.5 | $>$ 500 | 16 | 10 | 4 (16) |

TABLE: Diffraction of S-waves by the unit sphere. Number of GMRES iterations for a fixed density of points per wavelength.



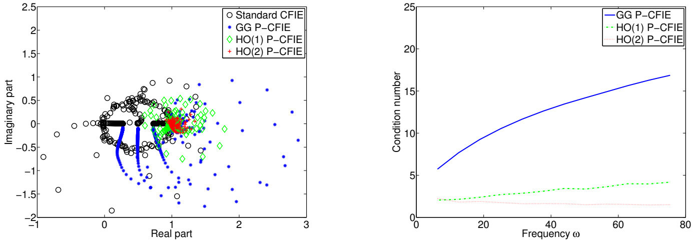
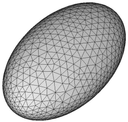
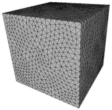



FIGURE: (**Left**) Distribution of the eigenvalues of the standard and different P-CFIEs ($\eta = 1$, $\kappa_s = 16\pi$ and $n_{\lambda_s} = 10$). (**Right**) Condition number with respect to $\omega$ ($n_{\lambda_s} = 10$)



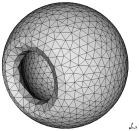
| #DOFs | $\omega$ | # iter CFIE ($\eta = 1$) | # iter GG P-CFIE | # iter HO(1) P-CFIE | # iter HO(2) P-CFIE |
|---|---|---|---|---|---|
| 3 594 | 25 | 48 | 8 | 8 | 7 (12) |
| 7 644 | 30 | 122 | 10 | 8 | 7 (13) |
| 41 310 | 60 | >500 | 11 | 9 | 8 (13) |
| 122 886 | 115 | >500 | 13 | 12 | 10 (15) |

TABLE: Diffraction of P-waves by an ellipsoid. Number of GMRES iterations for a fixed density of points per wavelength.



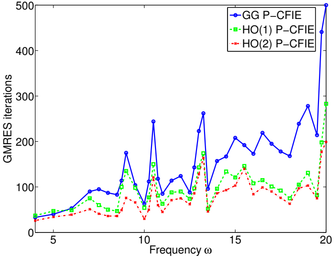
| #DOFs | $\omega$ | # iter CFIE ($\eta = 1$) | # iter GG P-CFIE | # iter HO(1) P-CFIE | # iter HO(2) P-CFIE |
|---|---|---|---|---|---|
| 1 446 | 2.5 | 14 | 10 | 9 | 9 (13) |
| 6 630 | 5 | 40 | 12 | 10 | 9 (13) |
| 26 505 | 11 | 120 | 13 | 10 | 9 (12) |
| 105 990 | 22 | >500 | 14 | 11 | 9 (13) |

TABLE: Diffraction of P-waves by a cube. Number of GMRES iterations for a fixed density of points per wavelength.



| #DOFs | $\omega$ | # iter GG P-CFIE | | # iter HO(1) P-CFIE | | # iter HO(2) P-CFIE | |
|---|---|---|---|---|---|---|---|
| 11 964 | 5 | 39 | 40 | 44 | 47 | 33 (54) | 34 (55) |
| 49 137 | 10 | 43 | 63 | 41 | 54 | 27 (42) | 30 (42) |
| 98 499 | 15 | 48 | 208 | 31 | 121 | 22 (28) | 103 (28) |
| 197 688 | 20 | 97 | $> 500$ | 48 | 283 | 36 (29) | 199 (29) |

TABLE: Diffraction of incident plane P-waves by the sphere with cavity. In each column, the first numbers give the number of GMRES iterations for $p_1 = d_1 = (-1, 0, 0)$ and the second numbers correspond to the incidence $p_2 = d_2 = (0, 0, 1)$.



- **Done : Dirichlet case**
  1. Utilization of the mathematical properties of the boundary integral operators to construct an efficient preconditioner
  2. Efficient combination of FM-BEM code & analytical preconditioner
  3. Better understanding of elastodynamic waves

- **Future work : Neumann case**
  1. Using $\mathbf{\Lambda}^{-1} = \left(\frac{1}{2}\mathrm{I} - P_0^{\varepsilon}(D)\right)^{-1} P_{-1}^{\varepsilon}(S)$ is not sufficient to construct well-conditioned BIEs for the Neumann scattering problem
  2. Idea : the optimal values of $\varepsilon_\gamma = 0.39\kappa_\gamma^{1/3}(\mathcal{H}^2)^{1/3}$ for $\gamma = s, p$ used in the regularisation of the square root $\kappa_{s,\varepsilon} = \kappa_s + i\varepsilon_s$ and $\kappa_{p,\varepsilon} = \kappa_s + i\varepsilon_p$ are not the good one.