Convergence to equilibrium for semigroups in Hilbert space

An intertwining approach

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Part I.
Abstract Framework

Contraction and Markov semigroups

Let $\mathcal{H}$ be a Hilbert space.

- We say $P = (P_t)_{t \geqslant 0}$ is a contraction semigroup if it is a C_0-semigroup on $\mathcal{H}$ and satisfies $|||P_t||| \leqslant 1$ for all $t \geqslant 0$.

Contraction and Markov semigroups

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- We say $P = (P_t)_{t \geqslant 0}$ is a contraction semigroup if it is a C_0-semigroup on $\mathcal{H}$ and satisfies $|||P_t||| \leqslant 1$ for all $t \geqslant 0$.

- We say P is an $\mathrm{L}^2(\mu)$-Markov semigroup if it is a Markov semigroup with invariant probability measure μ, i.e.

$$\mu P_t f = \mu f := \int f \; d\mu.$$

Convergence to equilibrium

Let P_∞ be the projection onto $\{f \in \mathcal{H} : P_t f = f \text{ for all } t \geqslant 0\}$.
Our aim is to understand the long-time behavior of

$$||P_t f - P_\infty f||_{\mathcal{H}} \to ?$$

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Definition 1

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$$\lim_{t\to\infty} ||P_t f - P_\infty f|| = 0.$$

Note: If $\mathbf{P}$ is another projection such that $||P_t f - \mathbf{P} f|| \to 0$, then by invariance $\mathbf{P} = P_\infty$.

Spectral gap inequality

P satisfies a spectral gap inequality $\mathrm{S}(\mathfrak{a})$ if

$$||P_t f - P_\infty f||_{\mathcal{H}} \leqslant e^{-\mathfrak{a}t} ||f - P_\infty f||_{\mathcal{H}}$$

where $\mathfrak{a} > 0$ is the spectral gap of P (i.e. the smallest such constant).

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- Bakry, Gentil, Ledoux's monograph [BGL14]
- Miclo [Mic15] self-adjoint, ergodic and hyperbounded $\implies$ spectral gap

Hypocoercivity & perturbed spectral gap

P satisfies a hypocoercive estimate $\mathrm{H}(C, \omega)$ if

$$||P_t f - P_\infty f||_{\mathcal{H}} \leqslant C e^{-\omega t} ||f - P_\infty f||_{\mathcal{H}},$$

where $C \geqslant 1$ and $\omega > 0$ are (in general, unknown) constants.

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- Villani's memoir [Vil09] (Lyapunov functional techniques)
- Mischler, Mouhot [MM16] (shrinking & enlarging spaces)
- Baudoin [Bau13] (Bakry-Émery theory & Γ-calculus)
- Hérau & Nier [HN04] (hypoellipticity techniques)

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P satisfies a perturbed spectral gap inequality $\mathrm{PS}(C,\omega)$ if it satisfies $\mathrm{H}(C,\omega)$ and ω is a gap in the point spectrum.

- Patie & Savov [PS16] (intertwining relationships).

Intertwining

Definition 2

Let P and Q be contraction semigroups on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$.

- We say that P intertwines with Q, written $P \lhd Q$, if there exists $\Lambda \in \mathbf{B}(\mathcal{K}, \mathcal{H})$ such that on $\mathcal{K}$ and for all $t \geqslant 0$ we have

$$P_t \Lambda = \Lambda Q_t .$$

- If $P \lhd Q$ and Λ is bijective then $P_t = \Lambda Q_t \Lambda^{-1}$. In this case we say P is in the similarity orbit of Q, and write $P \bowtie Q$.

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Literature:

- Sz.-Nagy & Foais [SNFBK10] (quasi-affinity)
- Douglas [Dou69] (unitary equivalence for normal operators)
- Dynkin [Dyn82] (Dynkin's criterion)
- Rogers & Pitman [RP80] (Brownian motion & Bessel process)

First convergence result

Theorem 1 (Patie, V. 2016)

Suppose $P \bowtie Q$ *and let* $\kappa(\Lambda) = |||\Lambda|||\ |||\Lambda^{-1}||| \geqslant 1$.

(i) *If* Q *satisfies* $H(C, \omega)$ *then* P *satisfies* $H(C\kappa(\Lambda), \omega)$.

(ii) *If* Q *satisfies* $S(\mathfrak{a})$ *then* P *satisfies* $PS(\kappa(\Lambda), \mathfrak{a})$.

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(i) *If Q satisfies $H(C,\omega)$ then P satisfies $H(C\kappa(\Lambda),\omega)$.*

(ii) *If Q satisfies $S(\mathfrak{a})$ then P satisfies $PS(\kappa(\Lambda),\mathfrak{a})$.*

Proof.

Let $P \bowtie Q$. Then, $P_\infty = \Lambda Q_\infty \Lambda^{-1}$. Since Q satisfies H(C,ω),

$$\begin{aligned}
||P_t f - P_\infty f||_{\mathcal{H}} &= ||\Lambda Q_t \Lambda^{-1} f - \Lambda Q_\infty \Lambda^{-1} f||_{\mathcal{H}} \\
&\leqslant |||\Lambda|||\ ||Q_t \Lambda^{-1} f - Q_\infty \Lambda^{-1} f||_{\mathcal{K}} \\
&\leqslant C|||\Lambda|||e^{-\omega t}||\Lambda^{-1} f - Q_\infty \Lambda^{-1} f||_{\mathcal{K}} \\
&= C|||\Lambda|||e^{-\omega t}||\Lambda^{-1} f - \Lambda^{-1} P_\infty f||_{\mathcal{K}} \\
&\leqslant C|||\Lambda|||\ |||\Lambda^{-1}|||e^{-\omega t}||f - P_\infty f||_{\mathcal{H}} \\
&= C\kappa(\Lambda)e^{-\omega t}||f - P_\infty f||_{\mathcal{H}}.
\end{aligned}$$

□

Beyond similarity via resolution of identity

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where E is the resolution of the identity for $-A_Q$, the generator of Q.

Writing $g = f - Q_\infty f$, we have

$$\begin{aligned} ||Q_t g||^2 &= \int_{\sigma(-A_Q)\setminus\{0\}} e^{-2\Re(\gamma)t} d\langle E_\gamma g, g\rangle \\ &\leqslant e^{-2\mathfrak{a}t} \int_{\sigma(-A_Q)\setminus\{0\}} d\langle E_\gamma g, g\rangle \\ &\leqslant e^{-\mathfrak{a}t} ||g||^2. \end{aligned}$$

Non-self-adjoint (unbounded) resolutions

From now on, suppose $P \triangleleft Q$ and the intertwining operator Λ satisfies

(a) $\mathrm{Ran}(\Lambda) \subset_d \mathcal{H}$, $(\overline{\mathrm{Ran}(\Lambda)} = \mathcal{H})$

(b) $\mathrm{Ker}(\Lambda \restriction_D) = 0$ where $\bigcup_{B \in \mathcal{B}(\mathbb{C})} \mathrm{Ran}(E_B) \subset \mathcal{D} \subset_d \mathcal{H}$,

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Then on $\mathcal{D}$ we can define, for $B \in \mathcal{B}(\mathbb{C})$,

$$F_B = \Lambda E_B \Lambda^{-1}$$

Each F_B is unbounded and non-self-adjoint.

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Proposition 1

Under the above assumptions, for each $f \in \mathcal{D}$ and all $t \geqslant 0$,

$$P_t f = \int_{\sigma(-A_Q)} e^{-\gamma t} dF_\gamma f.$$

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Proposition 1

Under the above assumptions, for each $f \in \mathcal{D}$ and all $t \geqslant 0$,

$$P_t f = \int_{\sigma(-A_Q)} e^{-\gamma t} dF_\gamma f.$$

As a corollary we deduce that for each $f \in \mathcal{D}$ and all $t \geqslant 0$,

$$||P_t f - P_\infty f||_{\mathcal{H}} \leqslant |||\Lambda||| e^{-\alpha t} \, ||\Lambda^{-1}(f - P_\infty f)||_{\mathcal{K}}.$$

Cannot be extended by density because Λ^{-1} is not necessarily bounded.

Integral representation of P

Assume further that there exists $m : \sigma(-A_Q) \to \mathbb{C}$ such that

(c) $F_\infty^m = \int_{\sigma(-A_Q)} m(\gamma) dF_\gamma$ is bounded on $\mathcal{H}$,

(d) for $t > T \geqslant 0$, $u(t) = \sup_{\Re(\gamma) \geqslant \mathfrak{a}} \frac{e^{-\Re(\gamma)t}}{|m(\gamma)|} < +\infty$.

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Theorem 2 (Patie, V. 2016)

Suppose Λ satisfies (a)–(d). Then for all $f \in \mathcal{H}$ and $t > T$,

(i) $P_t f = \int_{\sigma(-A_Q)} \frac{e^{-\gamma t}}{m(\gamma)} dF_\gamma^m f$,

(ii) $||P_t f - P_\infty f||_{\mathcal{H}} \leqslant |||F_\infty^m||| u(t) ||f - P_\infty f||_{\mathcal{H}}$,

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Remark 1

- If the sup is attained at $\mathfrak{a}$ then we have hypocoercivity estimate with spectral explanation for the constant, which is the spectral gap of Q.
- F_∞^m is no longer a resolution of the identity.
- Some sufficient conditions can be obtained when $P \triangleleft Q$ and $Q \triangleleft P$.

Proof of Theorem 2

The proof of (i) uses Proposition 1, condition (c) and the density of $\mathcal{D}$.

For the proof of (ii), writing $g = f - P_\infty f$, we have

$$\begin{aligned} ||P_t g||^2 &= \int_{\sigma(-A_Q)\setminus\{0\}} \frac{e^{-\gamma t}}{m(\gamma)} d\langle F_\gamma^m g, P_t g\rangle \\ &\leqslant u(t) \int_{\sigma(-A_Q)\setminus\{0\}} d\langle F_\gamma^m g, P_t g\rangle \\ &\leqslant |||F_\infty^m||| u(t)\, ||g||\, ||P_t g|| \end{aligned}$$

Part II.
Generalized Laguerre semigroups

Characterization

Let $\mathcal{N}$ denote the set of all functions $\Psi : i\mathbb{R} \to \mathbb{C}$ given by

$$\Psi(z) = \sigma^2 z^2 + mz + \int_{\mathbb{R}} (e^{zy} - 1 - zy)\Pi(dy),$$

where $\sigma^2, m \geqslant 0$ and Π is a measure satisfying $\int_{\mathbb{R}}(|x| \wedge x^2)\Pi(dx) < \infty$.

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$P = (e^{-tA})_{t\geqslant 0} \in \mathcal{GL}$, the set of generalized Laguerre semigroups, if

$$\mathcal{M}_{Af}(z+1) = -\Psi(-z)\mathcal{M}_f(z) + (z+1)\mathcal{M}_f(z+1), \quad z \in i\mathbb{R},$$

with $\mathcal{M}_f(s) = \int_0^\infty x^{s-1} f(x)dx$ the Mellin transform of f.

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There is a bijection between $\mathcal{N}$ and $\mathcal{GL}$. One can show that for suitable functions f,

$$\begin{aligned} -Af(x) &= \sigma^2 x f''(x) + \left(m + \sigma^2 - x\right) f'(x) \\ &\quad + \int_{\mathbb{R}} (f(e^y x) - f(x) - yxf'(x)) \frac{\Pi(dy)}{x}. \end{aligned}$$

Every $P \in \mathcal{GL}$ is an $\mathsf{L}^2(\mu)$-Markov semigroup with state space $(\mathbb{R}_+, \mathcal{B})$ and absolutely continuous, invariant probability measure μ.

Properties and classical Laguerre

Every $P \in \mathcal{GL}$ is an $\mathsf{L}^2(\mu)$-Markov semigroup with state space $(\mathbb{R}_+, \mathcal{B})$ and absolutely continuous, invariant probability measure μ.

P is self-adjoint on $\mathsf{L}^2(\mu)$ if and only if $\Pi \equiv 0$.

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If $\sigma = 1$, $\Pi \equiv 0$ we get the classical Laguerre semigroup of order $m \geqslant 0$, denoted by $Q^{(m)}$, which admits a spectral expansion

$$Q_t^{(m)} f = \sum_{n=0}^{\infty} e^{-nt} \langle f, e_n^{(m)} \rangle e_n^{(m)}.$$

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$$Q_t^{(m)} f = \sum_{n=0}^{\infty} e^{-nt} \langle f, e_n^{(m)} \rangle e_n^{(m)}.$$

From this we deduce that,

$$||Q_t^{(m)} f - Q_\infty^{(m)} f||_{\mathsf{L}^2(\mu)} \leqslant e^{-t} ||f - Q_\infty^{(m)} f||_{\mathsf{L}^2(\mu)}.$$

The similarity case

Theorem 3 (Patie, V. 2016)

Let $P \in \mathcal{GL}$ and suppose that the associated $\Psi \in \mathcal{N}$ satisfies

$$\Psi(z) = \Psi(-z) \iff m = 0 \quad \text{and} \quad \Pi(dx)\mathbb{1}_{\{x<0\}} = \Pi(dx)\mathbb{1}_{\{x>0\}}.$$

Then $P \bowtie Q^{(0)}$, where the intertwining operator Λ is a Mellin convolution operator with explicit Mellin multiplier. Furthermore,

$$||P_t f - P_\infty f||_{L^2(\mu)} \leqslant \kappa(\Lambda) e^{-t} ||f - P_\infty f||_{L^2(\mu)}.$$

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Remark 2

- When $\Pi \not\equiv 0$ then P is non-local and non-self-adjoint.
- The generator may or may not have a diffusive component.

Hypocoercivity for generalized Laguerre

Wiener-Hopf factorization: $\Psi(z) = -\phi_-(z)\phi_+(-z)$, $z \in i\mathbb{R}$, where

$$\phi_\pm(z) = \mathsf{k}_\pm + \mathsf{d}_\pm z + \int_0^\infty (1 - e^{-zy})\eta_\pm(dy) = \mathsf{k}_\pm + \mathsf{d}_\pm z + \int_0^\infty e^{-zy}\overline{\eta}_\pm(y)dy$$

$\mathsf{k}_\pm, \mathsf{d}_\pm \geqslant 0$ and $\int_0^\infty (1 \wedge y)\eta_\pm(dy) < \infty$.

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$\mathrm{k}_\pm, \mathrm{d}_\pm \geqslant 0$ and $\int_0^\infty (1 \wedge y)\eta_\pm(dy) < \infty$.

Theorem 4 (Patie, V. 2016)

Let $P \in \mathcal{GL}$. If Ψ is such that $\sigma > 0$ and $\overline{\eta}_- \overset{0}{\sim} \overline{\eta}_+$, then

$$||P_t f - P_\infty f||_{L^2(\mu)} \leqslant Ce^{-t}||f - P_\infty f||_{L^2(\mu)}$$

holds for all $f \in L^2(\mu)$ and $t \geqslant 0$, where $C > 1$ is explicit.

Sketch of proof

Key idea: $P \triangleleft Q^{(a)}$ and $Q^{(b)} \triangleleft P$ with $0 \leqslant a < b$, i.e.

$$P_t \Lambda = \Lambda Q_t^{(a)} \quad \text{and} \quad \tilde{\Lambda} P_t = Q_t^{(b)} \tilde{\Lambda}.$$

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The two intertwinings allow us to conclude that

$$F_\gamma = \sum_{n \leqslant \gamma} \langle \Lambda^{-1} f, e_n^{(a)} \rangle \Lambda e_n^{(a)} = \sum_{n \leqslant \gamma} \langle f, V_n \rangle P_n,$$

where $V_n, P_n \in \mathrm{L}^2(\mu)$.

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where $V_n, P_n \in \mathrm{L}^2(\mu)$. The function m is given by

$$m(n) = \frac{\Gamma(b+1)}{\Gamma(a+1)} \frac{\Gamma(n+a+1)}{\Gamma(n+b+1)},$$

which behaves asymptotically like n^{a-b}.

Conclusions

- Intertwining seems to be a fruitful approach for investigating convergence to equilibrium.
- Ideas can be applied in a general framework (independent of state space).
- Useful for transferring (spectral) information from known to unknown objects.

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How far can one go with intertwining?

Thank you for your time and attention

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