# Stochastic PDEs in the space of Tempered distributions

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Assumptions and conventions:

- All vector spaces are considered with $\mathbb{R}$ as the ground field.
- $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$: filtered complete probability space satisfying the usual conditions.
- Adapted processes and stopping times will be considered with respect to this filtration.
- We only consider continuous processes.

# A description of the problem

We have some Hilbert spaces $\mathbb{H}$ (Hermite-Sobolev spaces) with $\mathcal{S}(\mathbb{R}^d) \subset \mathbb{H} \subset \mathcal{S}'(\mathbb{R}^d)$.
We consider a class of SPDEs in $\mathbb{H}$ of the form

$$dY_t = A^*(Y_t).dB_t + L^*(Y_t)dt; \quad Y_0 = y \in \mathbb{H},$$

where $A^*, L^*$ are some (linear, unbounded) differential operators and $\{B_t\}$ is a finite dimensional standard Brownian motion.

$$\left.\begin{array}{c}\text{Solutions of finite dimensional SDEs}\\ \text{Duality arguments}\end{array}\right\} \implies \text{Existence of solutions of SPDEs.}$$

$$\text{Monotonicity inequality} \implies \text{Uniqueness of solutions of SPDEs.}$$

# A description of the problem contd.

Taking expectation on both sides of the SPDE leads to the existence of solution of

$$\frac{\partial \psi(t)}{\partial t} = L^* \psi(t); \quad \psi(0) = y.$$

Monotonicity inequality $\implies$ Uniqueness of solutions of SPDEs.

These results are proved for $y \in \mathcal{L}^1(\mathbb{R}^d)$ (where $\mathcal{L}^1(\mathbb{R}^d) \subset \mathbb{H}$, for an appropriate $\mathbb{H}$) and were motivated by results of [Rajeev and Thangavelu(2008)][1], where the initial conditions were taken as compactly supported distributions in $\mathbb{R}^d$.

[1] B. Rajeev and S. Thangavelu. *Probabilistic representations of solutions of the forward equations*, Potential Anal., vol. 28, no. 2, pp. 139–162, 2008.

# Outline

1. Schwartz Space with Hilbertian Topology
   - Hilbertian Topology and Hermite-Sobolev Spaces
2. Known results
   - Heat Equation
   - Forward Equations
   - Monotonicity inequality
3. New results
   - Ornstein-Uhlenbeck diffusion
   - Solution to SPDEs
   - Deterministic dependence on the initial condition
   - Solution to SPDEs Contd.

# Outline

$\mathcal{S}(\mathbb{R}^d)$ is the space of smooth rapidly decreasing $\mathbb{R}$-valued functions on $\mathbb{R}^d$. For the moment let us consider the case $d = 1$.

- The Schwartz topology (say $\tau$) on $\mathcal{S} = \mathcal{S}(\mathbb{R})$ is given by the semi-norms

$$|\phi|_{m,n} := \sup_t |t^m \phi^{(n)}(t)|, \; m, n = 0, 1, 2, \ldots.$$

- Let $\mathcal{S}'$ be the dual of $\mathcal{S}$. Elements of $\mathcal{S}'$ are called *tempered distributions*.

[2] Kiyosi Itô, *Foundations of stochastic differential equations in infinite-dimensional spaces*, volume 47 of *CBMS-NSF Regional Conference Series in Applied Mathematics*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1984.

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- Let $\mathcal{S}'$ be the dual of $\mathcal{S}$. Elements of $\mathcal{S}'$ are called *tempered distributions*.

We now describe a Hilbertian topology on $\mathcal{S}(\mathbb{R}^d)$. Main reference: [Itô(1984)][2].

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- Recall that an ONB for the Hilbert space $\mathcal{L}^2(\mathbb{R})$ is given by the Hermite functions

$$h_n(t) = (2^n n! \sqrt{\pi})^{-\frac{1}{2}} \exp\left(-\frac{t^2}{2}\right) H_n(t),\ n \geq 0$$

where $H_n(t)$ are the Hermite polynomials, which arise as the coefficients of $x^n$ in the expansion of $\exp(2xt - x^2)$. Note that $h_n \in \mathcal{S}$ and $\mathcal{S} \subset \mathcal{L}^2(\mathbb{R})$.

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- Denote the $\mathcal{L}^2$-norm and inner product by $\|\cdot\|$ and $\langle \cdot\,,\,\cdot \rangle$ respectively. For $\phi, \psi \in \mathcal{L}^2$, $p \in \mathbb{R}$, consider

$$\|\phi\|_p^2 :\ = \sum_{n=0}^{\infty} (2n+1)^{2p} \langle \phi\,,\, h_n \rangle^2,$$

$$\langle \phi\,,\, \psi \rangle_p := \sum_{n=0}^{\infty} (2n+1)^{2p} \langle \phi\,,\, h_n \rangle \langle \psi\,,\, h_n \rangle.$$

- Note that $\|\phi\|_p < \infty$ for $\phi \in \mathcal{S}$, $p \in \mathbb{R}$ and this gives a norm on $\mathcal{S}$ for every $p \in \mathbb{R}$. The corresponding inner product is given by $\langle \cdot\,,\,\cdot \rangle_p$.

- The completion of $(\mathcal{S}, \|\cdot\|_p)$ is a separable Hilbert space, denoted by $(\mathcal{S}_p, \|\cdot\|_p)$. These are the Hermite-Sobolev spaces.

[3] B. Rajeev, *From Tanaka's formula to Ito's formula: distributions, tensor products and local times*, in *Séminaire de Probabilités, XXXV*, volume 1755 of *Lecture Notes in Math.*, pages 371–389. Springer, Berlin, 2001.

- The completion of $(\mathcal{S}, \|\cdot\|_p)$ is a separable Hilbert space, denoted by $(\mathcal{S}_p, \|\cdot\|_p)$. These are the Hermite-Sobolev spaces.
- The Schwartz topology $\tau$ on $\mathcal{S}$ coincides with the countably Hilbertian topology determined by $\|\cdot\|_p$, $p = 1, 2, 3, \ldots$. For proof, refer to [Rajeev(2001)][3].
- We can similarly discuss $\mathcal{S}(\mathbb{R}^d)$, where we use

$$h_{n_1,n_2,..,n_d}(t_1, t_2, .., t_d) := \prod_{i=1}^{d} h_{n_i}(t_i)$$

instead of $h_n$.

[3] B. Rajeev, *From Tanaka's formula to Ito's formula: distributions, tensor products and local times*, in *Séminaire de Probabilités, XXXV*, volume 1755 of *Lecture Notes in Math.*, pages 371–389. Springer, Berlin, 2001.

- $(\mathcal{S}_{-p}, \|\cdot\|_{-p})$ is dual to $(\mathcal{S}_p, \|\cdot\|_p)$ for $p \geq 0$.
- $\mathcal{S}_0 = \mathcal{L}^2(\mathbb{R}), \mathcal{S} = \bigcap_{p\in\mathbb{R}} \mathcal{S}_p$, $\mathcal{S}' = \bigcup_{p\in\mathbb{R}} \mathcal{S}_p$.

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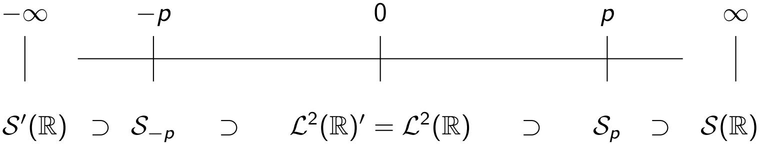

- Given a tempered distribution $\psi \in \mathcal{S}'(\mathbb{R}^d)$, the partial derivatives of $\psi$ are defined via the following relation

$$\langle \partial_i \psi \, , \, \phi \rangle := - \langle \psi \, , \, \partial_i \phi \rangle \, , \forall \phi \in \mathcal{S}(\mathbb{R}^d).$$

- $\partial_i : \mathcal{S}_p(\mathbb{R}^d) \to \mathcal{S}_{p-\frac{1}{2}}(\mathbb{R}^d)$ is a bounded linear operator. So the Laplacian $\triangle = \sum_{i=1}^d \partial_i^2$ is a bounded linear operator from $\mathcal{S}_p(\mathbb{R}^d)$ to $\mathcal{S}_{p-1}(\mathbb{R}^d)$.

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- For $x \in \mathbb{R}^d$, define translation operators on $\mathcal{S}(\mathbb{R}^d)$ by

$$(\tau_x \phi)(y) := \phi(y - x), \forall y \in \mathbb{R}^d.$$

We can extend this operator to $\tau_x : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ by

$$\langle \tau_x \phi \,,\, \psi \rangle := \langle \phi \,,\, \tau_{-x} \psi \rangle \,, \forall \phi \in \mathcal{S}'(\mathbb{R}^d), \psi \in \mathcal{S}(\mathbb{R}^d).$$

- $\tau_x : \mathcal{S}_p(\mathbb{R}^d) \to \mathcal{S}_p(\mathbb{R}^d)$ is a bounded linear operator.

Proposition ([Rajeev and Thangavelu(2008)])

*The Dirac distributions* $\delta_x \in \mathcal{S}_{-p}(\mathbb{R}^d)$ *for* $p > \frac{d}{4}$ *and there exists a constant* $C = C(p)$ *such that* $\|\delta_x\|_{-p} \leq C, \forall x \in \mathbb{R}^d$.

- Note that $\tau_x \delta_0 = \delta_x, x \in \mathbb{R}^d$.

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- Note that $\tau_x \delta_0 = \delta_x, x \in \mathbb{R}^d$.
- Multiplication of a distribution by a real valued smooth function $f$: $\langle M_f \psi\, ,\, \phi \rangle := \langle \psi\, ,\, f\phi \rangle\, , \forall \phi \in \mathcal{S}$. It is known that $M_{x_i} : \mathcal{S}_p(\mathbb{R}^d) \to \mathcal{S}_{p-\frac{1}{2}}(\mathbb{R}^d), i = 1, \cdots, d$ is a bounded linear operator.

# Outline

- Consider the Heat equation with initial condition $\bar{\phi} \in \mathcal{S}_p(\mathbb{R}^d)$ (for some $p \in \mathbb{R}$).

$$\partial_t \phi(t) = \frac{1}{2} \triangle \phi(t),\ t \leq T;\ \phi(0) = \bar{\phi}.$$

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- By an $\mathcal{S}_p(\mathbb{R}^d)$ valued solution of the previous equation, we mean an $\mathcal{S}_p(\mathbb{R}^d)$ valued continuous map on $[0, T]$, viz $t \mapsto \phi(t)$ such that the following equation holds in $\mathcal{S}_{p-1}(\mathbb{R}^d)$

$$\phi(t) = \bar{\phi} + \int_0^t \frac{1}{2} \triangle \phi(s)\, ds,\ t \leq T.$$

**Theorem ([Rajeev and Thangavelu(2003)])**

*The Heat equation has a unique $\mathcal{S}_p(\mathbb{R}^d)$ valued solution $\phi(t)$ given by*

$$\phi(t) = \mathbb{E}(\tau_{B_t} \bar{\phi}),$$

*where $\{B_t\}$ is a $d$ dimensional standard Brownian motion.*[a]

[a] B. Rajeev and S. Thangavelu, *Probabilistic representations of solutions to the heat equation*, Proc. Indian Acad. Sci. Math. Sci., vol. 113, no. 3, pp. 321–332, 2003.

Main reference: [Rajeev and Thangavelu(2008)]

- Let $\mathcal{F}$ be the Borel $\sigma$-field on $\Omega = C([0,\infty), \mathbb{R}^r)$, the space of $\mathbb{R}^r$ valued continuous functions on $[0,\infty)$.
- Let $P$ denote the Wiener measure.
- Under $P$, the process $B_t(\omega) := \omega(t)$, $\omega \in \Omega, t \geq 0$ is a standard $r$ dimensional Brownian Motion.
- Consider $\sigma = (\sigma_{ij}), i = 1, \cdots, d; j = 1, \cdots, r$ and $b = (b_1, \cdots, b_d)$ where $\sigma_{ij}, b_i$ are $C^\infty$ functions on $\mathbb{R}^d$ with bounded derivatives.

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Let $\{X(t,x)\}$ denote the unique strong solution on $(\Omega, \mathcal{F}, P)$ of the SDE

$$dX_t = \sigma(X_t).dB_t + b(X_t)dt; \quad X_0 = x.$$

A 'diffeomorphic modification' of $\{X(t,x)\}$ exists ([Kunita(1997)][4]).

### Theorem

*There exists a process* $\{\widetilde{X}(t,x)\}_{t\geq 0,\, x\in\mathbb{R}^d}$ *such that*

- *For all* $x \in \mathbb{R}^d$, $P(\widetilde{X}(t,x,\omega) = X(t,x,\omega),\ t \geq 0) = 1$.
- $P(x \mapsto \widetilde{X}(t,x,\omega)$ *is a diffeomorphism,* $\forall t \geq 0) = 1$.
- *(Flow property) Let* $\theta_t : \Omega \to \Omega$ *be the shift operator defined by* $(\theta_t\omega)(s) := \omega(s+t)$. *Then for* $s,t \geq 0$ *we have*

$$\widetilde{X}(t+s,x,\omega) = \widetilde{X}(t,\widetilde{X}(t,x,\omega),\theta_t\omega)$$

*for all* $x \in \mathbb{R}^d$, *a.s.* $\omega$.

[4] Hiroshi Kunita, *Stochastic flows and stochastic differential equations*, volume 24 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, 1997.

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In what follows, $\{X(t,x)\}$ will denote the modification obtained above.

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- Define $X_t(\omega) : C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$ by $(X_t(\omega)\phi)(x) := \phi(X(t,x,\omega))$. It is a continuous linear map.
- Let $\mathcal{E}'(\mathbb{R}^d)$ denote the space of compactly supported distributions (dual of $C^\infty(\mathbb{R}^d)$).
- Let $X_t(\omega)^* : \mathcal{E}'(\mathbb{R}^d) \to \mathcal{E}'(\mathbb{R}^d)$ be the transpose of the map $X_t(\omega)$. Note that

$$\langle X_t(\omega)^*\psi \, , \, \phi\rangle = \langle \psi \, , \, X_t(\omega)\phi\rangle \, , \forall \phi \in C^\infty(\mathbb{R}^d), \psi \in \mathcal{E}'(\mathbb{R}^d).$$

Define $Y_t(\omega) : \mathcal{E}'(\mathbb{R}^d) \to \mathcal{E}'(\mathbb{R}^d)$ by

$$Y_t(\omega)(\psi) := \sum_{|\alpha|\leq N} (-1)^{|\alpha|} \sum_{|\gamma|\leq|\alpha|} \int_V g_\alpha(x)$$
$$P_\gamma \left( (\partial^{\beta_1} X_1, \cdots, \partial^{\beta_d} X_d)_{|\beta^i|\leq|\alpha|} \right) (t, x, \omega) \, \partial^\gamma \delta_{X(t,x,\omega)} \, dx,$$

where $P_\gamma$ are some polynomials.

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where $P_\gamma$ are some polynomials.

### Theorem

*Let $\psi \in \mathcal{E}'(\mathbb{R}^d)$. There exists $p > 0$ such that $\{Y_t(\psi)\}$ is an $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued continuous adapted process and a.s.*

$$Y_t(\psi) = X_t^*(\psi),\ t \geq 0.$$

## Operators $A, L, A^*, L^*$

Now define the operators $A : C^\infty(\mathbb{R}^d) \to \mathcal{L}(\mathbb{R}^r, C^\infty(\mathbb{R}^d))$ and $L : C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$ as follows: for $\psi \in C^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

$$\begin{cases} A\phi := (A_1\phi, \cdots, A_r\phi), \\ A_i\phi(x) := \sum_{k=1}^d \sigma_{ki}(x)\partial_k\phi(x), \\ L\phi(x) := \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^t)_{ij}(x)\partial_{ij}^2\phi(x) + \sum_{i=1}^d b_i(x)\partial_i\phi(x). \end{cases}$$

We define the adjoint operators $A^* : \mathcal{E}'(\mathbb{R}^d) \to \mathcal{L}(\mathbb{R}^r, \mathcal{E}'(\mathbb{R}^d))$ and $L^* : \mathcal{E}'(\mathbb{R}^d) \to \mathcal{E}'(\mathbb{R}^d)$ as follows: for $\psi \in \mathcal{E}'(\mathbb{R}^d)$

$$\begin{cases} A^*\psi := (A_1^*\psi, \cdots, A_r^*\psi), \\ A_i^*\psi := -\sum_{k=1}^d \partial_k\left(\sigma_{ki}\psi\right), \\ L^*\psi := \frac{1}{2}\sum_{i,j=1}^d \partial_{ij}^2\left((\sigma\sigma^t)_{ij}\psi\right) - \sum_{i=1}^d \partial_i\left(b_i\psi\right). \end{cases}$$

### Estimates on $A^*, L^*$ [Rajeev and Thangavelu(2008)]

Fix $p > 0$ and $q > [p] + 4$. Then there exists constants $C_1(p) > 0, C_2(p) > 0$ such that for $\psi \in \mathcal{E}'(\mathbb{R}^d) \cap \mathcal{S}_{-p}(\mathbb{R}^d)$

$$\sum_{i=1}^{r} \|A_i^* \psi\|_{-q}^2 \leq C_1(p)\|\psi\|_{-p}^2, \quad \|L^*\psi\|_{-q} \leq C_2(p)\|\psi\|_{-p}.$$

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### Theorem ([Rajeev and Thangavelu(2008)])

*Fix $\psi \in \mathcal{E}'(\mathbb{R}^d)$. The $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued continuous adapted process $\{Y_t(\psi)\}$ satisfies the following equation in $\mathcal{S}_{-q}(\mathbb{R}^d)$ a.s.*

$$Y_t(\psi) = \psi + \underbrace{\int_0^t A^*(Y_s(\psi)).\, dB_s}_{=\sum_{i=1}^r \int_0^t A_i^*(Y_s(\psi))\, dB_s^i} + \int_0^t L^*(Y_s(\psi))\, ds, \; \forall t \geq 0.$$

Theorem ([Rajeev and Thangavelu(2008)])

*Fix $p > 0$ and $q > [p] + 4$. Let $\bar{\psi} \in \mathcal{E}'(\mathbb{R}^d) \cap \mathcal{S}_{-p}(\mathbb{R}^d)$. Then $\psi(t) := \mathbb{E}\, Y_t(\bar{\psi})$ solves*

$$\psi(t) = \bar{\psi} + \int_0^t L^* \psi(s)\, ds$$

*in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$.*
*Moreover the solution is unique if the pair $(A^*, L^*)$ satisfies the Monotonicity inequality, viz*

$$2 \langle \phi\, ,\, L^* \phi \rangle_{-q} + \sum_{i=1}^{r} \|A_i^* \phi\|_{-q}^2 \leq C\, \|\phi\|_{-q}^2,\ \forall \phi \in \mathcal{E}'(\mathbb{R}^d) \cap \mathcal{S}_{-p}(\mathbb{R}^d),$$

*where $C = C(p)$ is a positive constant.*

[5] R. J. DiPerna and P.-L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, in Invent. Math., vol. 98, no 3, pp. 511–547, 1989.

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Remark: When $\sigma = 0$, the PDE considered above reduces to linear transport equations considered in [DiPerna and Lions(1989)][5].

[5] R. J. DiPerna and P.-L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, in Invent. Math., vol. 98, no 3, pp. 511–547, 1989.

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[6] N. V. Krylov and B. L. Rozovskiĭ, *Stochastic evolution equations*, in *Current problems in mathematics, Vol. 14 (Russian)*, pages 71–147

[7] L. Gawarecki, V. Mandrekar, and B. Rajeev, *Linear stochastic differential equations in the dual of a multi-Hilbertian space*, in *Theory Stoch. Process.*, vol. 14, no 2, pp. 28–34, 2008

[8] B. L. Rozovskiĭ, *Stochastic evolution systems*, volume 35 of *Mathematics and its Applications (Soviet Series)*, Kluwer Academic Publishers Group, Dordrecht, 1990.

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- Proved in [Gawarecki, Mandrekar, and Rajeev(2009)][9] when $A^*, L^*$ were constant coefficient differential operators on $\mathcal{S}'(\mathbb{R}^d)$.

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# Motivation

The key observation is that if $\{Y_t\}$ solves an SDE of the form in $\mathcal{S}_p(\mathbb{R}^d)$

$$dY_t = A(Y_s).\, dB_s + L(Y_s)\, ds$$

then

$$E\|Y_t\|_p^2 \le \|Y_0\|_p^2 + E\int_0^t \underbrace{\left[2\left\langle Y_s\,,\, LY_s\right\rangle_p + \sum_{i=1}^r \|A_i(Y_s)\|_p^2\right]}_{\text{LHS of Monotonicity Inequality}} ds.$$

If above LHS of Monotonicity Inequality $\le C\|Y_s\|_q^2$, then Gronwall's Inequality alongwith $Y_0 = 0$ will give the uniqueness.

Let $\sigma = (\sigma_{ij})$ be a constant $d \times r$ matrix and $b = (b_1, ..., b_d) \in \mathbb{R}^d$.

Theorem ([Gawarecki, Mandrekar, and Rajeev(2009)])

*For every* $p \in \mathbb{R}, \exists$ *a constant* $C = C(p, d, (\sigma_{ij}), (b_j)) > 0$*, such that*

$$2 \langle \phi \, , \, L\phi \rangle_p + \sum_{i=1}^{r} \|A_i \phi\|_p^2 \leq C.\|\phi\|_p^2, \, \forall \phi \in \mathcal{S}(\mathbb{R}^d).$$

*Furthermore, by density arguments the above inequality can be extended to all* $\phi \in \mathcal{S}_{p+1}(\mathbb{R}^d)$.

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*Furthermore, by density arguments the above inequality can be extended to all $\phi \in \mathcal{S}_{p+1}(\mathbb{R}^d)$.*

Remark

*Monotonicity inequality holds for $(A^*, L^*)$ when $\sigma, b$ are as above.*

We consider the case $r = d$.

Theorem ([Bhar and Rajeev(2015)])

*Let* $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{R}^d$ *and* $C = (c_{ij})$ *be a real square matrix of order* $d$*. Let* $\sigma$ *be a constant function, i.e.* $\sigma(x) \equiv (\sigma_{ij}), \forall x \in \mathbb{R}^d$ *where* $\sigma_{ij} \in \mathbb{R}, i, j = 1, \cdots, d$*. Let* $b = (b_1, \cdots, b_d)$ *with* $b(x) := \alpha + C\,x, \forall x \in \mathbb{R}^d$*. Fix* $p \in \mathbb{R}$*. Then*[a]

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*Let $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{R}^d$ and $C = (c_{ij})$ be a real square matrix of order $d$. Let $\sigma$ be a constant function, i.e. $\sigma(x) \equiv (\sigma_{ij})$, $\forall x \in \mathbb{R}^d$ where $\sigma_{ij} \in \mathbb{R}$, $i, j = 1, \cdots, d$. Let $b = (b_1, \cdots, b_d)$ with $b(x) := \alpha + C\,x$, $\forall x \in \mathbb{R}^d$. Fix $p \in \mathbb{R}$. Then*[a]

1. *The maps $A_i^*$ are bounded linear operators from $\mathcal{S}_{p+\frac{1}{2}}(\mathbb{R}^d)$ to $\mathcal{S}_p(\mathbb{R}^d)$ and $L^*$ is a bounded linear operator from $\mathcal{S}_{p+1}(\mathbb{R}^d)$ to $\mathcal{S}_p(\mathbb{R}^d)$.*
2. *Monotonicity inequality for $A^*, L^*$ holds, i.e. there exists a positive constant $R = R(p, d, (\sigma_{ij}), (b_j))$, such that*

$$2\,\langle \phi\,,\, L^*\phi \rangle_p + \sum_{i=1}^{d} \|A_i^*\phi\|_p^2 \leq R\,\|\phi\|_p^2$$

*for all $\phi \in \mathcal{S}_{p+1}(\mathbb{R}^d)$.*

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# Outline

## Ornstein-Uhlenbeck diffusion

Consider the case $\sigma = I, b(x) = -x$.

$$dX_t = dB_t - X_t\,dt; \quad X_0 = x$$

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Note: $x \mapsto X(t,x,\omega)$ is an affine map and hence is a $C^\infty$ function with bounded derivatives.

- Define a continuous linear map, denoted by $X_t(\omega) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ and given by
$$(X_t(\omega)\phi)(x) := \phi(X(t,x,\omega)),\, x \in \mathbb{R}^d.$$
- Let $X_t^*(\omega) : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ denote the transpose of the map $X_t(\omega)$. Then for any $\psi \in \mathcal{S}'(\mathbb{R}^d)$,
$$\langle X_t^*(\psi)\,,\,\phi\rangle = \langle \psi\,,\,X_t(\phi)\rangle\,,\,\forall \phi \in \mathcal{S}(\mathbb{R}^d).$$

- Fix $\psi \in \mathcal{L}^1(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$. The identification is given by

$$\phi \mapsto \int_{\mathbb{R}^d} \phi(x)\psi(x)\, dx.$$

In fact $\mathcal{L}^1(\mathbb{R}^d) \subset \mathcal{S}_{-p}(\mathbb{R}^d)$ for any $p > \frac{d}{4}$.

- Define $Y_t(\omega)(\psi) := \int_{\mathbb{R}^d} \psi(x)\delta_{X(t,x,\omega)}\, dx$.

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- Define $Y_t(\omega)(\psi) := \int_{\mathbb{R}^d} \psi(x)\delta_{X(t,x,\omega)}\, dx$.
- $Y_t(\psi)$ is a well-defined element of $\mathcal{S}_{-p}(\mathbb{R}^d)$ for any $p > \frac{d}{4}$.
- $\mathbb{E}\,\|Y_t(\psi)\|_{-p}^2 \leq C^2 \left(\int_{\mathbb{R}^d} |\psi(x)|\, dx\right)^2 < \infty$ for some constant $C > 0$.
- Observe that

$$\begin{aligned}\langle Y_t(\psi)\,,\,\phi\rangle &= \int_{\mathbb{R}^d} \psi(x)\,\phi(X(t,x))\, dx\\ &= \int_{\mathbb{R}^d} \psi(x)\,(X_t(\phi))(x)\, dx = \langle \psi\,,\,X_t(\phi)\rangle\,,\ \forall \phi \in \mathcal{S}(\mathbb{R}^d).\end{aligned}$$

- $Y_t(\psi) = X_t^*(\psi)$.

## Theorem ([Bhar(2016)])

*Let $p > \frac{d}{4}$ and $\psi \in \mathcal{L}^1(\mathbb{R}^d)$. Then*[a] *the $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued continuous adapted process $\{Y_t(\psi)\}$ satisfies the following equation in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$, a.s.*

$$Y_t(\psi) = \psi + \int_0^t A^*(Y_s(\psi))\, dB_s + \int_0^t L^*(Y_s(\psi))\, ds, \; \forall t \geq 0.$$

*This solution is also unique.*

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[a] Suprio Bhar, *Characterizing Gaussian flows arising from Itô's stochastic differential equations*, Potential Analysis, pages 1–17, 2016. doi: 10.1007/s11118-016-9578-6.

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### Sketch of Proof.

By Itô's formula for any $\phi \in \mathcal{S}(\mathbb{R}^d)$, and any $x \in \mathbb{R}^d$

$$\begin{aligned}(X_t(\phi))(x) = \phi(X(t,x)) &= \phi(x) + \int_0^t A\phi(X(s,x)).\, dB_s + \int_0^t L\phi(X(s,x))\, ds \\ &= \phi(x) + \int_0^t (X_s(A\phi))(x).\, dB_s + \int_0^t (X_s(L\phi))(x)\, ds\end{aligned}$$

## Sketch of Proof (contd.)

Then for $\phi \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned}
\langle Y_t(\psi)\,,\,\phi\rangle &= \langle \psi\,,\,X_t(\phi)\rangle \\
&= \left\langle \psi\,,\,\phi + \int_0^t X_s(A\phi).\,dB_s + \int_0^t X_s(L\phi)\,ds \right\rangle \\
&= \langle \psi\,,\,\phi\rangle + \int_0^t \langle \psi\,,\,X_s(A\phi)\rangle\,.\,dB_s + \int_0^t \langle \psi\,,\,X_s(L\phi)\rangle\;ds \\
&= \langle \psi\,,\,\phi\rangle + \int_0^t \langle A^* Y_s(\psi)\,,\,\phi\rangle\,.\,dB_s + \int_0^t \langle L^* Y_s(\psi)\,,\,\phi\rangle\;ds \\
&= \left\langle \psi + \int_0^t A^* Y_s(\psi).\,dB_s + \int_0^t L^* Y_s(\psi)\,ds\,,\,\phi \right\rangle
\end{aligned}$$

## Sketch of Proof (contd.)

Then for $\phi \in \mathcal{S}(\mathbb{R}^d)$,

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&= \langle \psi\,,\,\phi\rangle + \int_0^t \langle A^* Y_s(\psi)\,,\,\phi\rangle\,.\,dB_s + \int_0^t \langle L^* Y_s(\psi)\,,\,\phi\rangle\;ds \\
&= \left\langle \psi + \int_0^t A^* Y_s(\psi).\,dB_s + \int_0^t L^* Y_s(\psi)\,ds\,,\,\phi \right\rangle
\end{aligned}$$

Proof of uniqueness: Gronwall's inequality + Monotonicity inequality

### Theorem (B.)

*Let $p > \frac{d}{4}$ and $\psi \in \mathcal{L}^1(\mathbb{R}^d)$. Then $\bar{\psi}(t) := \mathbb{E}\, Y_t(\psi)$ solves the equation*

$$\frac{d}{dt}\bar{\psi} = L^*\bar{\psi},$$

*i.e. the equality*

$$\mathbb{E}\, Y_t(\psi) = \psi + \int_0^t L^*\left(\mathbb{E}\, Y_s(\psi)\right)\, ds$$

*holds in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$. Furthermore this is the unique solution.*

Consider random fields which arise as solutions of SDEs:

$$dX_t = \sigma(X_t)\,dB_t + b(X_t)\,dt,\ \forall t \geq 0; \quad X_0 = X$$

where the coefficients $\sigma = (\sigma_{ij}), b = (b_i), 1 \leq i, j \leq d$ are Lipschitz continuous and the random variable $X$ is independent of the Brownian motion $\{B_t\}$. For any $x \in \mathbb{R}^d$, let $\{X_t^x\}$ denote the solution of the SDE with $X_0 = x$.

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It is known that the solutions to such equations are Gaussian if $X$ is Gaussian (or a constant), $\sigma$ is a constant $d \times d$ matrix and $b(x) = a + bx, \forall x \in \mathbb{R}^d$ for some $a \in \mathbb{R}^d, b \in \mathbb{R}$.

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The strong solutions of the above equations are maps $F : [0,\infty) \times \mathbb{R}^d \times C([0,\infty), \mathbb{R}^d) \to \mathbb{R}^d$ such that the solutions with initial value $X$ and Brownian motion $\{B_t\}$ is given at time $t$ by

$$X_t = F(t, X, B), \text{ a.s..}$$

In the Gaussian case as above (i.e. $\sigma$ is constant and $b(x) = a + bx$) it is known that a.s.

$$F(t, x, B) = e^{tb}x + (e^{tb} - 1)b^{-1}a + \int_0^t e^{(t-s)b}\sigma \, dB_s.$$

We wish to characterize the maps $F$ for which the solutions of the above SDEs are Gaussian.

We make a definition of the class of SDEs such that $\{X_t^x\}$ has a deterministic 'local' component.

### Definition

*We say the general solution of the SDE depends deterministically on the initial condition, if there exists a function* $f : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ *such that for any* $x \in \mathbb{R}^d$*, we have a.s.*

$$X_t^x(\omega) = f(t, x) + X_t^0(\omega),\ t \geq 0.$$

Remark: In this case, for every fixed $x \in \mathbb{R}^d$, the map $t \mapsto \frac{\partial f}{\partial t}(t, x) = (\frac{\partial f_1}{\partial t}(t, x), \cdots, \frac{\partial f_d}{\partial t}(t, x))$ is continuous.

## Theorem ([Bhar(2016)])

*Let $\sigma, b$ be Lipschitz continuous functions. Suppose the following happen*[a]*:*

1. *there exists an $x \in \mathbb{R}^d$ such that the determinant of $(\sigma_{ij}(x))$ is not zero,*
2. $b_i \in C^1(\mathbb{R}^d, \mathbb{R}), i = 1, \cdots, d$ *where* $b = (b_1, \cdots, b_d)$,
3. *for every fixed $x \in \mathbb{R}^d$, the map $t \in [0, \infty) \mapsto \frac{\partial f}{\partial t}(t, x)$ is of bounded variation.*

*Then the general solution of the SDE depends deterministically on the initial condition if and only if $\sigma$ is a real non-singular matrix of order $d$ and $b$ is of the form $b(x) = \alpha + Cx$ and $f(t, x) = e^{tC}x$ where $\alpha \in \mathbb{R}^d$ and $C$ is a real square matrix of order $d$.*

[a] Suprio Bhar, *Characterizing Gaussian flows arising from Itô's stochastic differential equations*, Potential Analysis, pages 1–17, 2016. doi: 10.1007/s11118-016-9578-6.

## Proposition (B.)

*Let $\sigma, b$ be Lipschitz continuous functions.*

1. *Suppose the general solution of the SDE depends deterministically on the initial condition, where the function $f$ has the decomposition $f(t,x) = g(t)h(x)$ with $g \in C^1([0,\infty),\mathbb{R}), h : \mathbb{R}^d \to \mathbb{R}^d$. Then $f(t,x) = \tilde{g}(t)x$ for some $\tilde{g} \in C^1([0,\infty),\mathbb{R})$ with $\tilde{g}(0) = 1$.*
2. *The solution to the SDE depends deterministically on the initial condition in the following form: for each $x \in \mathbb{R}^d$, a.s. $t \geq 0$*

$$X_t^x = g(t)x + X_t^0,$$

*for some $g \in C^1([0,\infty),\mathbb{R})$ with $g(0) = 1$ if and only if $\sigma$ is a constant $d \times d$ matrix, $b(x) = \alpha + \beta x$ and $g(t) = e^{\beta t}$, $t \geq 0$ where $\alpha \in \mathbb{R}^d, \beta \in \mathbb{R}$. In this case, the solution has the form*

$$X_t^x = \begin{cases} e^{\beta t}x + \sigma \int_0^t e^{\beta(t-s)}\,dB_s + \frac{e^{\beta t}-1}{\beta}\alpha, \text{ if } \beta \neq 0 \\ x + t\alpha + \sigma B_t, \text{ if } \beta = 0. \end{cases}$$

- Let $\sigma = (\sigma_{ij})$ be a real square matrix of order $d$.
- Let $b = (b_1, \cdots, b_d)$ be of the form $b(x) = \alpha + Cx$ where $\alpha \in \mathbb{R}^d$ and $C$ is a real square matrix of order $d$.

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- Define the continuous linear maps $X_t(\omega) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ and $X_t^*(\omega) : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$.
- For $\psi \in \mathcal{L}^1(\mathbb{R}^d)$, define $Y_t(\psi)$ as before. Then $Y_t(\psi) = X_t^*(\psi)$.

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### Theorem (B.)

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$$Y_t(\psi) = \psi + \int_0^t A^*(Y_s(\psi))\, dB_s + \int_0^t L^*(Y_s(\psi))\, ds, \forall t \geq 0.$$

*This is also the unique solution of the previous equation. Furthermore,*

$$\mathbb{E}\, Y_t(\psi) = \psi + \int_0^t L^* \mathbb{E}\, Y_s(\psi)\, ds$$

*holds in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$. Furthermore this is the unique solution.*

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# Thank You