Rate of convergence for monotone approximations of non-local Isaacs' equations

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Non-local operator

- A prototypical example of non-local operator is the "fractional laplace" operator and is defined as

$$(-\triangle)^{\sigma/2}u(x) = C_{n,\sigma}\ P.V \int_{\mathbb{R}^N} \frac{u(x) - u(x+z)}{|z|^{N+\sigma}}\ dz, \tag{1}$$

for $\sigma \in (0,2)$. $C_{n,\sigma}$ is a constant depending on n and σ, given by

$$C_{n,\sigma} = \left(\int_{\mathbb{R}^N} \frac{1-\cos(z_1)}{|z|^{N+\sigma}}\ dz \right)^{-1}. \tag{2}$$

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- The fractional Laplacian ($(-\triangle)^{\sigma/2}$) could also be thought of as a Pseudo-differential operator with symbol $|\xi|^{\sigma}$. In other words, in terms of Fourier transform,

$$(-\triangle)^{\sigma/2}\phi = \mathcal{F}^{-1}\left(|.|^{\sigma}\mathcal{F}(\phi)\right), \tag{3}$$

for any function ϕ for which the right hand side makes sense.

- A general non-local operator would be of the following form:

$$Lu(x) = -\int_{\mathbb{R}^N} [u(x+z) - u(x) - z.\nabla u(x)\mathbf{1}_{|z|<1}(z)]\ \nu(dz),$$

where, ν is a non-negative Radon measure on $\mathbb{R}^N$ with possible singularity at origin satisfying the condition

$$\int_{\mathbb{R}^N} \min\{|z|^2, 1\}\ \nu(dz) < \infty. \tag{4}$$

This Radon measures are often referred to as the Lévy measure as there associate non-local operators turn out to be the generator of pure jump Lévy processes.

Non-local Isaacs' Equation

We are interested in the initial value problem

$$u_t + \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \Big\{ -f^{\alpha,\beta}(t,x) + c^{\alpha,\beta}(t,x) u(t,x) \\ - b^{\alpha,\beta}(t,x).\nabla u(t,x) - \mathcal{I}^{\alpha,\beta}[u](t,x) \Big\} = 0 \quad \text{in } Q_T, \qquad \text{(N.Is.Eq)}$$

$$u(0,x) = u_0(x) \quad \text{in } \mathbb{R}^N, \qquad \text{(I.C)}$$

where $Q_T := (0, T] \times \mathbb{R}^N$ and α, β respectively takes values in two metric spaces $\mathcal{A}$ and $\mathcal{B}$.

Non-local Isaacs' Equation

The operator $\mathcal{I}^{\alpha,\beta}$ signifies the nonlocality of the equation and is defined as

$$\mathcal{I}^{\alpha,\beta}[\phi](t,x) := \int_{\mathbb{R}^M \setminus \{0\}} \big(\phi(t, x + \eta^{\alpha,\beta}(t,x;z)) - \phi(t,x) \\ - \eta^{\alpha,\beta}(t,x;z).\nabla_x \phi(t,x)\big)\, \nu(dz) \quad (\mathrm{N.T})$$

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- The quantity ν is a so-called Lévy measure on $\mathbb{R}^M\setminus\{0\}$ i.e a nonnegative Radon measure with a possible singularity at the origin.
- The functions $\eta^{\alpha,\beta}(t,x;z)$ are defined from $[0,\infty)\times\mathbb{R}^N\times\mathbb{R}^M$ to $\mathbb{R}^N$ and it is possible to have $\eta^{\alpha,\beta}(t,x;z) = 0$ on a set of positive measure. This makes the equation (N.Is.Eq) degenerate.

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- We know that Hamilton-Jacobi-Bellman(HJB) equations have direct corresponds to the optimal control problems.
- Instead of a single control when we will deal with two controls, one wants to maximize and the other wants to minimize the cost function then it called Differential Game problem.
- Let us consider the controlled state dynamics with a pure jump process on a filtered probability space $(\Omega_t, \mathcal{F}_t, P_t, \mathcal{F}_{t,\cdot})$ as

$$dX(s) = b(s, X(s), A(s), B(s))\ ds \\ + \int_{\mathbb{R}^M \setminus \{0\}} \eta(s, X(s_-), A(s), B(s); w) \tilde{N}(ds, dw),$$

where $s \in (t, T]$. $X(\cdot)$ satisfies the initial condition $X(t) = x$ where, $x \in \mathbb{R}^N$.

Contd...

$A(s)$ and $B(s)$ are two predictable control processes with values in $\mathcal{A}$ and $\mathcal{B}$ respectively.

- The pay-off functional is given by

$$J(t, x; A, B) = E^{t,x}\left[\int_t^T f(s, X(s), A(s), B(s))\, ds + g(X(T))\right].$$

- Strategy Of the Game: Let two players 'Player I' and 'Player II' are playing the game. Player I chooses the control process $A(\cdot)$ and it want to minimize the pay-off functional. At the same time, Player II chooses the control process $B(\cdot)$ which wants to maximize the pay-off functional.

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- It has a great application in mathematical-finance. For example, it appears in financial market where the prices follow a 'jump process' and two market participants are competing to optimize their respective interest.

Viscosity Solutions

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Theorem ((Lipschitz Solution) Jakobsen and Karlsen '05,'06)

Under suitable condition on the coefficients of the equation (N.Is.Eq), we have

(a) *There exists a unique bounded viscosity solution u of the initial value problem (N.Is.Eq).*

(b) *The viscosity solution u is Lipschitz continuous with the the following inequity :*

$$||u(t,.)||_{C^{0,1}(\mathbb{R}^N)} \leq K(tC + ||u_0||_{C^{0,1}(\mathbb{R}^N)}).$$

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- The study of numerical approximation in the context of viscosity solutions began in the early eighties. The numerical approximations of the classical first order HJB equations of the form

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- The rate of convergence for classical first order Bellman-Isaacs equations is showed to be $\frac{1}{2}$ and it is the optimal rate.

- Error estimates of the numerical approximation for second order HJB equation

$$u_t + H(t, x, u, \nabla u, D^2 u) = 0$$

remained unsolved for a number of years, although the convergence of such schemes have already been established by Souginidis and Barles (1991).

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  - Finally, Krylov settled the issue for a class of monotone schemes for convex Hamiltonian cases in his series of papers (1997,2000,2005) by introducing the method of shaking the coefficient.
  - In a parallel development, Barles & Jakobsen offered a general approach, based on Krylov's method, to estimate general monotone approximation scheme to a convex second order HJB equation.

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  - On space dimension greater than one, partial results have been obtained with uniform ellipticity, we refer the initial works by Caffarelli & Souganidis(2008,2010) where existence of an algebraic rate of convergence is established.
  - Later there have been some extensions of these results but those are restricted to the uniformly elliptic equations only. Also these results were unable to address the optimal rate of convergence.

Available Results for non local equations:

Study of numerical approximation of non-local Bellman-Isaacs equations is a more recent one. The studies were concentrated on the equations of the form

$$u_t + \sup_{\alpha \in \mathcal{A}} \{ -f^\alpha(t,x) + c^\alpha(t,x)u(t,x) - tr[a^\alpha(t,x)D^2 u(x)]$$
$$- b^\alpha(t,x).\nabla u(t,x) - \mathcal{I}^\alpha[u](t,x) \} = 0 \quad \text{in } Q_T,$$
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- Karlsen, Jakobsen & Biswas have studied the monotone approximate schemes for such convex non-local Bellman equations in there series of papers (2007,2008,2010) where the specific emphasis was given in estimating the rate of convergence.
- The approaches are based on the method of shaking the coefficients and hence not applicable to the equations of type (N.Is.Eq).

Method of Shaking the coefficient

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- Let u be a solution (weak or numerical) of

$$u_t + \sup_{\alpha}\{f^{\alpha}(t,x) + c^{\alpha}(t,x)u - b^{\alpha}(t,x).\nabla u - \frac{1}{2}tr(a^{\alpha}(t,x)D^2u)\} \leq 0. \tag{Conv.Eq}$$

- We further consider an $\varepsilon > 0$ and u^{ε} be the solution(weak or numerical)

$$u_t^{\varepsilon} + \sup_{\alpha, y \in \mathcal{A}\times B_1}\{f^{\alpha}(t,x+\varepsilon y) + c^{\alpha}(t,x+\varepsilon y)u^{\varepsilon} - b^{\alpha}(t,x+\varepsilon y).\nabla u^{\varepsilon} - \frac{1}{2}tr(a^{\alpha}(t,x+\varepsilon y)D^2u^{\varepsilon})\} \leq 0.$$

We note that if y is singleton then $u^{\varepsilon}(t,x) = u(t,x+\varepsilon y)$.

- Let us take a non negative function ξ such that $\xi \in C_c^{\infty}(B_1)$ and have unit integral. We define

$$u^{(\varepsilon)}(t,x) = \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N} u(t,y)\xi\left(t, \frac{x-y}{\varepsilon}\right)dy.$$

The above inequality implies, for every $\alpha \in \mathcal{A}$ and $y \in B_1$

$$u_t^\varepsilon(t, x - \varepsilon y) + \{f^\alpha(t, x) + c^\alpha(t, x) u^\varepsilon(t, x - \varepsilon y)$$
$$- b^\alpha(t, x).\nabla u^\varepsilon(t, x - \varepsilon y) - \frac{1}{2} tr(a^\alpha(t, x) D^2 u^\varepsilon(t, x - \varepsilon y)\} \leq 0. \quad \text{(Sh.Eq)}$$

- Now we multiply (Sh.Eq) by ξ and integrate over y, we finally get

$$u_t^{\varepsilon(\varepsilon)} + \sup_\alpha \{f^\alpha(t, x) + c^\alpha(t, x) u^{\varepsilon(\varepsilon)} - b^\alpha(t, x).\nabla u^{\varepsilon(\varepsilon)} - \frac{1}{2} tr(a^\alpha(t, x) D^2 u^{\varepsilon(\varepsilon)})\} \leq 0.$$

  Hence, we observe that we are able to construct a smooth solution of (Conv.Eq) from the given particular solution u. Then the properties of mollifiers, consistency of numerical scheme with smooth solution and comparison principle would help us to get the desired rate of convergence.

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- Hence, once we consider the non-convex Hamiltonian ; this method will no longer work.

Questions: To construct a consistent monotone numerical approximate scheme of equation (N.Is.Eq), i.e,

$$u_t + \inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} \Big\{ -f^{\alpha,\beta}(t,x) + c^{\alpha,\beta}(t,x)u(t,x) - b^{\alpha,\beta}(t,x).\nabla u(t,x) - \mathcal{I}^{\alpha,\beta}[u](t,x) \Big\} = 0 \quad \text{in } Q_T,$$

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and to find the rate of convergence of the scheme with the viscosity solutions.

* These are the joint work of I*, Imran and Jakobsen.

Assumptions

We now list the set of working assumptions. These are necessary for the wellposedness and regularity results for the problem (N.Is.Eq).

(A.1) The control sets $\mathcal{A}, \mathcal{B}$ are separable metric spaces. Moreover, the functions $f^{\alpha,\beta}(t,x), c^{\alpha,\beta}(t,x), b^{\alpha,\beta}(t,x)$ and $\eta^{\alpha,\beta}(t,x;z)$ are continuous in α, β, t, x and z.

(A.2) There exist a constant $K > 0$ such that for every α, β ,

$$||u_0||_1 + ||f^{\alpha,\beta}||_1 + ||c^{\alpha,\beta}||_1 + ||b^{\alpha,\beta}||_1 \leq K.$$

(A.3) For $x, y \in \mathbb{R}^N$ and $z \in \mathbb{R}^M$ we have

$$|\eta(t,x;z) - \eta(t,y;z)| \leq C|z|\,|x-y| \quad \text{and,} \quad |\eta(t,x;z)| \leq C_0\,|z|.$$

(A.4) The Lévy measure $\nu(dz)$ is Radon measure on $\left(\mathbb{R}^M, \mathcal{B}(\mathbb{R}^M)\right)$ and it has density $k(z)$ of the form

$$0 \leq k(z) \leq C\frac{e^{-\Lambda|z|}}{|z|^{M+\sigma}}$$

for some $\sigma \in (0,2)$ and $\Lambda > 0$.

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In the first case as $\sigma < 1$ we can easily verify that the Lévy measure should satisfy the following:

$$\begin{aligned}\int_{|z|<1} |z|\nu(dz) &< C \int_{|z|<1} |z| \frac{e^{-\Lambda|z|}}{|z|^{M+\sigma}}\, dz \\ &\leq C \int_{|z|<1} \frac{1}{|z|^{M+\sigma-1}}\, dz < \infty.\end{aligned}$$

The above estimate would not be true for $\sigma \geq 1$.

Monotone scheme for $\sigma \in (0,1)$.

Let $\Delta t, \Delta x > 0$ be the discretisation parameters/ mesh size in the time and spatial variables, respectively. We use the notation h to denote the vector $\big(\Delta t, \Delta x\big)$. we consider $M = \frac{T}{\Delta t}$ and write the mesh as

$$\mathcal{G}_h^N = \Big\{(t_n, x_{\mathbf{m}}) : t_n = n\Delta t, x_{\mathbf{m}} = \mathbf{m}\,\Delta x;\ \mathbf{m} \in \mathbb{Z}^N, n = 0, 1, ..., M\Big\}.$$

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- The equation (N.Is.Eq) has two main spatial component the need discretisation. The local term $b^{\alpha,\beta}(t,x).\nabla u(t,x)$ and the nonlocal term $\mathcal{I}^{\alpha,\beta}[u](t,x)$.

- The discretisation of $b^{\alpha,\beta}(t,x).\nabla u(t,x)$ is denoted by $\mathcal{D}_h^{\alpha,\beta}[u](t,x)$ and defined as

$$\mathcal{D}_h^{\alpha,\beta}[u](t,x) = \sum_{i=1}^{N}\Big[b_i^{\alpha,\beta,+}(t,x)\frac{u(t,x+e_i\Delta x)-u(t,x)}{\Delta x} + b_i^{\alpha,\beta,-}(t,x)\frac{u(t,x-e_i\Delta x)-u(t,x)}{\Delta x}\Big],$$

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where where $e_i \in \mathbb{R}^N$ are the unit normal vectors in $\mathbb{R}^N$.

- We rewrite $\mathcal{D}_h^{\alpha,\beta}[u](t,x)$ as

$$\mathcal{D}_h^{\alpha,\beta}[u](t,x) = \sum_{\mathbf{j}\neq 0} d_{h,\mathbf{j},}^{\alpha,\beta}(t,x) \Big[\phi(t_n, x + x_{\mathbf{j}}) - \phi(t_n, x) \Big],$$

where, $d_{h,\pm e_i}^{\alpha,\beta}(t,x) = \frac{b^{\alpha,\beta,\pm}(t,x)}{\Delta x}$ and $d_{h,\mathbf{j}}^{\alpha,\beta}(t,x) = 0$ otherwise.

- The next step is to propose a suitable discritisation of the non-local term $\mathcal{I}^{\alpha,\beta}[u]$. We use the notation $\mathcal{I}_h^{\alpha,\beta}[u]$ denote the quadrature based approximation of $\mathcal{I}^{\alpha,\beta}[u]$ based on the spatial grid $\Delta x\,\mathbb{Z}^N$.

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  - Let i_h be the piece-wise linear/multilinear/affine interpolation operator based on the spatial grid. Then the operator i_h has the following form

$$i_h[\phi](x) = \sum_{\mathbf{j} \in \mathbb{Z}^N} \phi(x_{\mathbf{j}}) \omega_{\mathbf{j}}(x; h) \quad \text{with,} \quad \omega_{\mathbf{j}}(x; h) \geq 0$$

for all $x \in \mathbb{R}^N$.

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    for all $x \in \mathbb{R}^N$.

  - In addition, the functions $\omega_{\mathbf{j}}(x;h)$ are Lipschitz continuous and $\omega_{\mathbf{j}}(x_{\mathbf{k}};h) = \delta_{\mathbf{j},\mathbf{k}}$ and $\sum_{\mathbf{j}} \omega_{\mathbf{j}}(x;h) = 1$.

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  - We also remark that $\omega_{\mathbf{j}}(0;h) = 0$ if $\mathbf{j} \neq 0$ and

$$\omega_{\mathbf{j}}(x;h) \leq \frac{K}{\Delta x}|x| \quad \text{for all} \quad x \in \mathbb{R}^N.$$

- The monotone discretization of $\mathcal{I}^{\alpha,\beta}$ is now given by

$$\mathcal{I}_h^{\alpha,\beta}[\varphi](t,x) = \sum_{\mathbf{j}\neq 0} \big(\varphi(t,x+x_{\mathbf{j}}) - \varphi(t,x)\big)\kappa_{\mathbf{j}}^{\alpha,\beta}(t,x;h),$$

where

$$\kappa_{\mathbf{j}}^{\alpha,\beta}(t,x;h) = \int_{\mathbb{R}^M\setminus\{0\}} \omega_{\mathbf{j}}(\eta^{\alpha,\beta}(t,x;z);h)\nu(dz)$$
$$\leq \frac{C}{\Delta x}\int_{\mathbb{R}^M\setminus\{0\}} \eta^{\alpha,\beta}(t,x;z)\nu(dz) \leq \frac{C}{\Delta x}\int_{\mathbb{R}^M\setminus\{0\}} |z|\nu(dz) \leq \frac{K}{\Delta x}$$

- The monotone discretization of $\mathcal{I}^{\alpha,\beta}$ is now given by

$$\mathcal{I}_h^{\alpha,\beta}[\varphi](t,x) = \sum_{\mathbf{j}\neq 0} \big(\varphi(t,x+x_{\mathbf{j}}) - \varphi(t,x)\big)\kappa_{\mathbf{j}}^{\alpha,\beta}(t,x;h),$$

where

$$\kappa_{\mathbf{j}}^{\alpha,\beta}(t,x;h) = \int_{\mathbb{R}^M\setminus\{0\}} \omega_{\mathbf{j}}(\eta^{\alpha,\beta}(t,x;z);h)\nu(dz)$$
$$\leq \frac{C}{\Delta x}\int_{\mathbb{R}^M\setminus\{0\}} \eta^{\alpha,\beta}(t,x;z)\nu(dz) \leq \frac{C}{\Delta x}\int_{\mathbb{R}^M\setminus\{0\}} |z|\nu(dz) \leq \frac{K}{\Delta x}$$

(Note that when $\sigma \in [1,2)$, we can not make sense of this quantity.)

- By the property of i_h we can conclude that there is number $K(N)$ depending only on N such that

$$\sum_{\mathbf{j}\neq 0} \kappa_{\mathbf{j}}^{\alpha,\beta}(t,x;h) \leq \frac{K(N)}{\Delta x}.$$

Fully discrete numerical scheme

The solution of the scheme on the grid $\mathcal{G}_h^N$ is denoted as U_h and defined as $U_{\mathbf{j}}^n = U_h(t_n, x_{\mathbf{j}})$ for any $(t_n, x_{\mathbf{j}}) \in \mathcal{G}_h^N$. Then, for two parameters $\theta, \vartheta \in [0,1]$, the implicit-explicit fully discrete scheme is written as

$$\begin{aligned} U_{\mathbf{j}}^n =& U_{\mathbf{j}}^{n-1} - \Delta t \inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} \Big\{ -\theta \mathcal{D}_h^{\alpha,\beta}[u]_{\mathbf{j}}^n - (1-\theta)\mathcal{D}_h^{\alpha,\beta}[U]_{\mathbf{j}}^{n-1} \\ & - f_{\mathbf{j}}^{\alpha,\beta,n-1} + c_{\mathbf{j}}^{\alpha,\beta,n} U_{\mathbf{j}}^{n-1} - \vartheta \mathcal{I}_h^{\alpha,\beta}[U]_{\mathbf{j}}^n - (1-\vartheta)\mathcal{I}_h^{\alpha,\beta}[U]_{\mathbf{j}}^{n-1} \Big\}, \end{aligned} \tag{Nu.Schm}$$

$$U_{\mathbf{j}}^0 = u(0, x_{\mathbf{j}}) \quad \text{for all } \mathbf{j} \in \mathbb{Z}^N.$$

In the above, for any generic function $\gamma^{\alpha,\beta}$, we have followed abbreviation $\gamma(t_n, x_{\mathbf{j}}) = \gamma_{\mathbf{j}}^n$ for $t_n \in \Delta t \times \{0, 1, 2, ..., \frac{T}{\Delta t}\}$ and any multiindex $\mathbf{j} \in \mathbb{Z}^N$.

Monotonicity of the scheme.

- We see that, the weight functions of the respective approximations $\mathcal{D}_h^{\alpha,\beta}$ and $\mathcal{I}_h^{\alpha,\beta}$ are non-negative.

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$$\kappa_{h,\bar{j}}^{\alpha,\beta,n} = \sum_{\mathbf{j}\neq 0} \kappa_{h,\mathbf{j},\bar{\mathbf{j}}}^{\alpha,\beta,n} \leq K_I(\Delta x)^{-1}$$

$$\text{and,} \quad d_{h,\bar{\mathbf{j}}}^{\alpha,\beta,n} = \sum_{\mathbf{j}\neq 0} d_{h,\mathbf{j},\bar{\mathbf{j}}}^{\alpha,\beta,n} \leq K_D(\Delta x)^{-1}.$$

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- Then we have the following lemma.

Lemma

The scheme (Nu.Schm) is monotone under the following condition: if $\Delta x \le 1$, along with

$$\frac{\Delta t}{\Delta x}\Big((1-\theta)K_D + (1-\vartheta)K_I\Big) + \Delta t \sup_{\alpha,\beta} |c^{\alpha,\beta}| \le 1. \qquad \text{(CFL)}$$

Wellpossedness of numerical scheme

Theorem

Assume (A.1)-(A.4) and the CFL condition (CFL) *hold. Then there exists unique bounded solution* U_h *of* (Nu.Schm). *Moreover, the scheme* (Nu.Schm) *is* L^∞*-stable, more specifically*
$|U_h| \leq e^{\sup_{\alpha,\beta}|c^{\alpha,\beta}|_0 t_n}\Big[|u_0| + \sup_{\alpha,\beta}|f^{\alpha,\beta}|\Big].$

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$|U_h| \leq e^{\sup_{\alpha,\beta} |c^{\alpha,\beta}|_0 t_n} \Big[|u_0| + \sup_{\alpha,\beta} |f^{\alpha,\beta}| \Big]$.

- The proof of the above theorem uses the classical fixed point argument with an induction on the time stepping.

Main result

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Theorem (I*,Imran, Jakobsen, 2016)

Let (A.1)-(A.4) be true with $\sigma \in (0,1)$ and u_0 be bounded and Lipschitz continuous function on $\mathbb{R}^N$. Furthermore, assume that the CFL condition (CFL) holds and U_h be the unique bounded function on $\mathcal{G}_h^N$ that solves (Nu.Schm). If $u \in C^{0,1}(\overline{Q_T})$ is the unique viscosity solution of (N.Is.Eq) in $\overline{Q_T}$, then there exists a constant $C > 0$ depending only on $||u_0||$, $||\nabla u_0||$ such that

$$|U_h - u| \leq C(\Delta t^{1/2} + \Delta x^{1/2}) \quad \text{on} \quad \mathcal{G}_h^N.$$

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- The proof uses the discretize version of the technique of doubling the variable adopted in non-local setup.
- Due to too many technical details we are skipping the details of the proof here.

Monotone scheme for $\sigma \in [1, 2)$.

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Monotone scheme for $\sigma \in [1, 2)$.

- In this case, we propose a two-step approximation method for (N.Is.Eq).
- The first step would be to approximate the equation by cutting-off the singularity of the Lévy measure at a suitable level. Let $\delta > 0$ be a positive constant and $\nu_\delta(dz) = \mathbf{1}_{|z|>\delta}(z)\,\nu(dz)$.We replace $\nu(dz)$ by $\nu_\delta(\,dz)$ in the non local term $\mathcal{I}^{\alpha,\beta}[\phi]$ and write

$$\begin{aligned}\mathcal{J}^{\alpha,\beta,\delta}[\phi](t,x) = \int_{|z|>\delta} \big(\phi(t, x + \eta^{\alpha,\beta}(t,x;z)) - \phi(t,x) \\ - \eta^{\alpha,\beta}(t,x;z)\cdot\nabla_x\phi(t,x)\big)\ \nu(dz) \\ := \mathcal{I}^{\alpha,\beta,\delta}[\phi](t,x) - b_\delta^{\alpha,\beta}(t,x)\cdot\nabla_x\phi(t,x),\end{aligned}$$

where

$$\mathcal{I}^{\alpha,\beta,\delta}[\phi](t,x) = \int_{|z|>\delta} \big(\phi(t, x + \eta^{\alpha,\beta}(t,x;z)) - \phi(t,x)\big)\,\nu(dz),$$

$$b_\delta^{\alpha,\beta}(t,x) = \int_{|z|>\delta} \eta^{\alpha,\beta}(t,x;z)\,\nu(dz).$$

- We now replace $\mathcal{I}^{\alpha,\beta}[u]$ in (N.Is.Eq) by $\mathcal{J}^{\alpha,\beta,\delta}[\phi]$ and obtain the perturbed problem as

$$u_t^\delta + \inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} \Big\{ -f^{\alpha,\beta}(t,x) + c^{\alpha,\beta}(t,x)u^\delta(t,x) - \tilde{b}_\delta^{\alpha,\beta}(t,x).\nabla u^\delta(t,x)$$
$$-\mathcal{I}^{\alpha,\beta,\delta}[u^\delta](t,x)\Big\} = 0 \quad \text{in } Q_T, \quad \text{(Pt.N.Is.Eq)}$$
$$u^\delta(0,x) = u_0(x) \quad \text{in } \mathbb{R}^N, \qquad \text{(Pt.I.C)}$$

where $\tilde{b}_\delta^{\alpha,\beta}(t,x) = b^{\alpha,\beta}(t,x) + b_\delta^{\alpha,\beta}(t,x)$.

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-\mathcal{I}^{\alpha,\beta,\delta}[u^\delta](t,x) \Big\} = 0 \quad \text{in } Q_T, \quad \text{(Pt.N.Is.Eq)} \\
u^\delta(0,x) = u_0(x) \quad \text{in } \mathbb{R}^N, \qquad \text{(Pt.I.C)}
\end{aligned}
$$

where $\tilde{b}_\delta^{\alpha,\beta}(t,x) = b^{\alpha,\beta}(t,x) + b_\delta^{\alpha,\beta}(t,x)$.

- The specific choice of ν_δ guarantees us to consider the monotone discretisation of $\tilde{b}_\delta(t,x)\cdot\nabla_x u(t,x)$ and $\mathcal{I}^{\alpha,\beta,\delta}[u](t,x)$ in a similar way as for the case $\sigma\in(0,1)$ and those discritisations are respectively denoted as $\mathcal{D}_h^{\alpha,\beta,\delta}[u](t,x)$ and $\mathcal{I}_h^{\alpha,\beta,\delta}[u](t,x)$.

Fully discrete scheme

- We now propose the following discretisation of (Pt.N.Is.Eq)-(Pt.I.C):

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\begin{aligned}
[U^\delta]_{\mathbf{j}}^n = & [U^\delta]_{\mathbf{j}}^{n-1} - \Delta t \inf_{\alpha\in\mathcal{A}} \sup_{\beta\in\mathcal{B}} \Big\{ -\theta \mathcal{D}_h^{\alpha,\beta,\delta}[U^\delta]_{\mathbf{j}}^n \\
& \quad - (1-\theta)\mathcal{D}_h^{\alpha,\beta,\delta}[U^\delta]_{\mathbf{j}}^{n-1} - f_{\mathbf{j}}^{\alpha,\beta,n} + c_{\mathbf{j}}^{\alpha,\beta,n}[U^\delta]_{\mathbf{j}}^{n-1} \\
& \quad - \vartheta \mathcal{I}_h^{\alpha,\beta,\delta}[U^\delta]_{\mathbf{j}}^n - (1-\vartheta)\mathcal{I}_h^{\alpha,\beta,\delta}[U^\delta]_{\mathbf{j}}^{n-1} \Big\}, \qquad \text{(Pt.Nu.Schm)} \\
[U^\delta]_{\mathbf{j}}^0 = & u(0,x_{\mathbf{j}}) \quad \text{for all } \mathbf{j}\in\mathbb{Z}^N, \qquad \text{(Pt.Nu.I)}
\end{aligned}
$$

where U_h^δ denotes the solution of this scheme on the grid $\mathcal{G}_h^N$ and defined as $U_h^\delta(t_n, x_{\mathbf{j}}) = [U^\delta]_{\mathbf{j}}^n$.

- This scheme is monotone under a suitable CFL condition. We note that this CFL condition would depend on the cut-off parameter δ.

Main Result

- Our aim is to estimate $||U_h^\delta - u^\delta||_{L^\infty(\mathcal{G}_h^N)}$ for every fixed $\delta > 0$. .

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- We can verify that the bounds on $\tilde{b}^{\alpha,\beta}$, $\mathcal{I}^{\alpha,\beta,\delta}$(two main terms of the perturbed equation) and their respective approximations would depend on

$$\int_{|z|>\delta} |z|\nu(dz) \approx \Gamma(\sigma,\delta) = \begin{cases} \delta^{1-\sigma} & \text{when,} \quad \sigma > 1 \\ -\log\delta & \text{when,} \quad \sigma = 1 \end{cases} \tag{5}$$

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  and hence this quantity $\Gamma(\sigma,\delta)$ will iterate to the proof of error estimate as well.
- We follow the similar steps as in the case of $\sigma \in (0,1)$ and finally, have the following error estimate for perturbed problem (Pt.N.Is.Eq).

Theorem (I*,Imran, Jakobsen, 2016)

Let (A.1)-(A.4) hold and u^δ be the unique viscosity solution of (Pt.N.Is.Eq)-(Pt.I.C). For very $\delta > 0$, if U_h^δ is the solution of the scheme (Pt.Nu.Schm)-(Pt.Nu.I) then

$$\sup_{(t,x)\in\mathcal{G}_h^N} |u^\delta(t,x) - U_h^\delta(t,x)| \le K_1(\Delta t + \Delta x)^{1/2}\Gamma(\sigma,\delta) + K_2(\Delta x)^{3/2}\delta^{-\sigma}. \tag{6}$$

Main result

- The final step would be to estimate the viscosity solution u of our main equation with the solution u^{δ} of perturbed equation. We invoke the continuous dependence estimate proved by Karlsen & Jakobsen(JDE,2005)

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$$
\begin{aligned}
|u(t,x) - u^\delta(t,x)| &\leq C T^{1/2} \sup_{\alpha,\beta} \sqrt{\int_{\mathbb{R}^N \setminus \{0\}} |\eta^{\alpha,\beta}(t,x,z)|^2 \; |\nu - \nu_\delta|(dz)} \\
&\leq K \delta^{1-\frac{\sigma}{2}}
\end{aligned} \tag{7}
$$

for $(t,x) \in Q_T$.

Main result

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$$\begin{aligned} |u(t,x) - u^\delta(t,x)| &\leq CT^{1/2} \sup_{\alpha,\beta} \sqrt{\int_{\mathbb{R}^N \setminus \{0\}} |\eta^{\alpha,\beta}(t,x,z)|^2 \, |\nu - \nu_\delta|(dz)} \\ &\leq K\delta^{1-\frac{\sigma}{2}} \end{aligned} \tag{7}$$

for $(t,x) \in Q_T$.

- Hence, for any $(t,x) \in \mathcal{G}_h^N$, we combine the estimates (6) and (7) obtain

$$|u(t,x) - U_h^\delta(t,x)| \leq K_1(\Delta t + \Delta x)^{1/2}\Gamma(\sigma,\delta) + K_2(\Delta x)^{3/2}\delta^{-\sigma} + K\delta^{1-\frac{\sigma}{2}}$$

for every $\delta > 0$.

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- Finally choose δ optimally (consider $\delta = (\Delta x + \Delta t)^{1/\sigma}$) to get the desired result.

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Theorem (I*,Imran, Jakobsen, 2016)

Let (A.1)-(A.4) hold and u be the unique viscosity solution of (N.Is.Eq)-(I.C) with $\sigma \in [1,2)$. If $\tilde{U}_h$ is the solution of the scheme (Pt.Nu.Schm)-(Pt.Nu.I) for $\delta = (\Delta x + \Delta t)^{1/\sigma}$, then

$$|u - \tilde{U}_h| \leq \begin{cases} K(\Delta t + \Delta x)^{\frac{1}{\sigma} - \frac{1}{2}} & \text{when, } \quad \sigma > 1 \\ K(\Delta t + \Delta x)^{\frac{1}{2}} |\log(\Delta t + \Delta x)| & \text{when, } \quad \sigma = 1 \end{cases} \tag{8}$$

in $\mathcal{G}_h^N$.

Remarks

- We have obtained the optimal rate of convergence for the case $\sigma < 1$.
- For the case $\sigma \geq 1$ the rates, we obtained, are depending on the order of the non local term σ.

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- For the case $\sigma \geq 1$ the rates, we obtained, are depending on the order of the non local term σ.

The immediate question that comes :

- Whether we can find a better rate of convergence for the case $\sigma \geq 1$ and what will be the optimal rate of convergence in this case?

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Thank YOU.