Stochastic PDEs in the space of Tempered distributions

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Assumptions and conventions:

- All vector spaces are considered with $\mathbb{R}$ as the ground field.
- $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$: filtered complete probability space satisfying the usual conditions.
- Adapted processes and stopping times will be considered with respect to this filtration.
- We only consider continuous processes.
A description of the problem

We have some Hilbert spaces $\mathbb{H}$ (Hermite-Sobolev spaces) with $S(\mathbb{R}^d) \subset \mathbb{H} \subset S'(\mathbb{R}^d)$.

We consider a class of SPDEs in $\mathbb{H}$ of the form

$$dY_t = A^*(Y_t).dB_t + L^*(Y_t)dt; \quad Y_0 = y \in \mathbb{H},$$

where $A^*, L^*$ are some (linear, unbounded) differential operators and $\{B_t\}$ is a finite dimensional standard Brownian motion.

Solutions of finite dimensional SDEs
Duality arguments

$\implies$ Existence of solutions of SPDEs.

Monotonicity inequality $\implies$ Uniqueness of solutions of SPDEs.
Taking expectation on both sides of the SPDE leads to the existence of solution of

\[
\frac{\partial \psi(t)}{\partial t} = L^* \psi(t); \quad \psi(0) = y.
\]

Monotonicity inequality \(\implies\) Uniqueness of solutions of SPDEs.

These results are proved for \(y \in \mathcal{L}^1(\mathbb{R}^d)\) (where \(\mathcal{L}^1(\mathbb{R}^d) \subset \mathcal{H}\), for an appropriate \(\mathcal{H}\)) and were motivated by results of [Rajeev and Thangavelu(2008)]\(^1\), where the initial conditions were taken as compactly supported distributions in \(\mathbb{R}^d\).

1. Schwartz Space with Hilbertian Topology
   - Hilbertian Topology and Hermite-Sobolev Spaces

2. Known results
   - Heat Equation
   - Forward Equations
   - Monotonicity inequality

3. New results
   - Ornstein-Uhlenbeck diffusion
   - Solution to SPDEs
   - Deterministic dependence on the initial condition
   - Solution to SPDEs Contd.
Outline

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$S(\mathbb{R}^d)$ is the space of smooth rapidly decreasing $\mathbb{R}$-valued functions on $\mathbb{R}^d$. For the moment let us consider the case $d = 1$.

- The Schwartz topology (say $\tau$) on $S = S(\mathbb{R})$ is given by the semi-norms
  \[ |\phi|_{m,n} = \sup_t |t^m \phi^{(n)}(t)|, \quad m, n = 0, 1, 2, \ldots \]

- Let $S'$ be the dual of $S$. Elements of $S'$ are called tempered distributions.

$\mathcal{S}(\mathbb{R}^d)$ is the space of smooth rapidly decreasing $\mathbb{R}$-valued functions on $\mathbb{R}^d$. For the moment let us consider the case $d = 1$.

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- Let $\mathcal{S}'$ be the dual of $\mathcal{S}$. Elements of $\mathcal{S}'$ are called \textit{tempered distributions}.

We now describe a Hilbertian topology on $\mathcal{S}^{\prime}(\mathbb{R}^d)$. Main reference: [Itô(1984)]\(^2\).

Recall that an ONB for the Hilbert space $L^2(\mathbb{R})$ is given by the Hermite functions

$$h_n(t) = (2^n n! \sqrt{\pi})^{-\frac{1}{2}} \exp\left(-\frac{t^2}{2}\right) H_n(t), \quad n \geq 0$$

where $H_n(t)$ are the Hermite polynomials, which arise as the coefficients of $x^n$ in the expansion of $\exp(2xt - x^2)$. Note that $h_n \in S$ and $S \subset L^2(\mathbb{R})$. 
Recall that an ONB for the Hilbert space $\mathcal{L}^2(\mathbb{R})$ is given by the Hermite functions

$$h_n(t) = (2^n n! \sqrt{\pi})^{-\frac{1}{2}} \exp \left( -\frac{t^2}{2} \right) H_n(t), \quad n \geq 0$$

where $H_n(t)$ are the Hermite polynomials, which arise as the coefficients of $x^n$ in the expansion of $\exp(2xt - x^2)$. Note that $h_n \in S$ and $S \subset \mathcal{L}^2(\mathbb{R})$.

Denote the $\mathcal{L}^2$-norm and inner product by $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$ respectively. For $\phi, \psi \in \mathcal{L}^2$, $p \in \mathbb{R}$, consider

$$\| \phi \|^2_p := \sum_{n=0}^{\infty} (2n + 1)^{2p} \langle \phi, h_n \rangle^2,$$

$$\langle \phi, \psi \rangle_p := \sum_{n=0}^{\infty} (2n + 1)^{2p} \langle \phi, h_n \rangle \langle \psi, h_n \rangle.$$

Note that $\| \phi \|^2_p < \infty$ for $\phi \in S$, $p \in \mathbb{R}$ and this gives a norm on $S$ for every $p \in \mathbb{R}$. The corresponding inner product is given by $\langle \cdot, \cdot \rangle_p$. 
The completion of \((S, \| \cdot \|_p)\) is a separable Hilbert space, denoted by \((S_p, \| \cdot \|_p)\). These are the Hermite-Sobolev spaces.

The completion of \((\mathcal{S}, \| \cdot \|_p)\) is a separable Hilbert space, denoted by \((\mathcal{S}_p, \| \cdot \|_p)\). These are the Hermite-Sobolev spaces.

The Schwartz topology \(\tau\) on \(\mathcal{S}\) coincides with the countably Hilbertian topology determined by \(\| \cdot \|_p, p = 1, 2, 3, \ldots\). For proof, refer to [Rajeev(2001)]\(^3\).

We can similarly discuss \(\mathcal{S}(\mathbb{R}^d)\), where we use

\[
h_{n_1, n_2, \ldots, n_d}(t_1, t_2, \ldots, t_d) = \prod_{i=1}^{d} h_{n_i}(t_i)
\]

instead of \(h_n\).

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• \((S_{-p}, \| \cdot \|_{-p})\) is dual to \((S_p, \| \cdot \|_p)\) for \(p \geq 0\).

• \(S_0 = \mathcal{L}^2(\mathbb{R}), S = \bigcap_{p \in \mathbb{R}} S_p, S' = \bigcup_{p \in \mathbb{R}} S_p\).
(\mathcal{S}_p, \| \cdot \|_p) is dual to (\mathcal{S}_p, \| \cdot \|_p) for p \geq 0.

S_0 = L^2(\mathbb{R}), S = \bigcap_{p \in \mathbb{R}} S_p, S' = \bigcup_{p \in \mathbb{R}} S_p.
Given a tempered distribution \( \psi \in \mathcal{S}'(\mathbb{R}^d) \), the partial derivatives of \( \psi \) are defined via the following relation

\[
\langle \partial_i \psi, \phi \rangle := -\langle \psi, \partial_i \phi \rangle, \quad \forall \phi \in \mathcal{S}(\mathbb{R}^d).
\]

\( \partial_i : \mathcal{S}_p(\mathbb{R}^d) \to \mathcal{S}_{p-\frac{1}{2}}(\mathbb{R}^d) \) is a bounded linear operator. So the Laplacian \( \Delta = \sum_{i=1}^{d} \partial_i^2 \) is a bounded linear operator from \( \mathcal{S}_p(\mathbb{R}^d) \) to \( \mathcal{S}_{p-1}(\mathbb{R}^d) \).
Given a tempered distribution \( \psi \in S'(\mathbb{R}^d) \), the partial derivatives of \( \psi \) are defined via the following relation

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\langle \partial_i \psi, \phi \rangle := -\langle \psi, \partial_i \phi \rangle, \quad \forall \phi \in S(\mathbb{R}^d).
\]

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For \( x \in \mathbb{R}^d \), define translation operators on \( S(\mathbb{R}^d) \) by

\[
(\tau_x \phi)(y) := \phi(y - x), \quad \forall y \in \mathbb{R}^d.
\]

We can extend this operator to \( \tau_x : S'(\mathbb{R}^d) \to S'(\mathbb{R}^d) \) by

\[
\langle \tau_x \phi, \psi \rangle := \langle \phi, \tau_{-x} \psi \rangle, \quad \forall \phi \in S'(\mathbb{R}^d), \psi \in S(\mathbb{R}^d).
\]

\( \tau_x : S_p(\mathbb{R}^d) \to S_p(\mathbb{R}^d) \) is a bounded linear operator.
Proposition ([Rajeev and Thangavelu(2008)])

The Dirac distributions \( \delta_x \in S_{-p}(\mathbb{R}^d) \) for \( p > \frac{d}{4} \) and there exists a constant \( C = C(p) \) such that \( \|\delta_x\|_{-p} \leq C \), \( \forall x \in \mathbb{R}^d \).

- Note that \( \tau_x \delta_0 = \delta_x, x \in \mathbb{R}^d \).
Proposition ([Rajeev and Thangavelu(2008)])

The Dirac distributions $\delta_x \in S_{-p}(\mathbb{R}^d)$ for $p > \frac{d}{4}$ and there exists a constant $C = C(p)$ such that $\|\delta_x\|_{-p} \leq C$, $\forall x \in \mathbb{R}^d$.

- Note that $\tau_x\delta_0 = \delta_x$, $x \in \mathbb{R}^d$.
- Multiplication of a distribution by a real valued smooth function $f$:
  $\langle M_f \psi, \phi \rangle := \langle \psi, f \phi \rangle$, $\forall \phi \in S$. It is known that $M_{x_i} : S_p(\mathbb{R}^d) \to S_{p-\frac{1}{2}}(\mathbb{R}^d)$, $i = 1, \cdots, d$ is a bounded linear operator.
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Consider the Heat equation with initial condition $\bar{\phi} \in S_p(\mathbb{R}^d)$ (for some $p \in \mathbb{R}$).

$$\partial_t \phi(t) = \frac{1}{2} \triangle \phi(t), \ t \leq T; \ \phi(0) = \bar{\phi}.$$
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By an \( S_p(\mathbb{R}^d) \) valued solution of the previous equation, we mean an \( S_p(\mathbb{R}^d) \) valued continuous map on \([0, T]\), viz \( t \mapsto \phi(t) \) such that the following equation holds in \( S_{p-1}(\mathbb{R}^d) \)

\[
\phi(t) = \bar{\phi} + \int_0^t \frac{1}{2} \triangle \phi(s) \, ds, \ t \leq T.
\]

**Theorem ([Rajeev and Thangavelu(2003)])**

The Heat equation has a unique \( S_p(\mathbb{R}^d) \) valued solution \( \phi(t) \) given by

\[
\phi(t) = \mathbb{E}(\tau_{B_t} \bar{\phi}),
\]

where \( \{B_t\} \) is a \( d \) dimensional standard Brownian motion.\(^a\)

Main reference: [Rajeev and Thangavelu(2008)]

- Let $\mathcal{F}$ be the Borel $\sigma$-field on $\Omega = C([0, \infty), \mathbb{R}^r)$, the space of $\mathbb{R}^r$ valued continuous functions on $[0, \infty)$.

- Let $P$ denote the Wiener measure.

- Under $P$, the process $B_t(\omega) := \omega(t), \omega \in \Omega, t \geq 0$ is a standard $r$ dimensional Brownian Motion.

- Consider $\sigma = (\sigma_{ij}), i = 1, \cdots, d; j = 1, \cdots, r$ and $b = (b_1, \cdots, b_d)$ where $\sigma_{ij}, b_i$ are $C^\infty$ functions on $\mathbb{R}^d$ with bounded derivatives.
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Let $\{X(t, x)\}$ denote the unique strong solution on $(\Omega, F, P)$ of the SDE

$$dX_t = \sigma(X_t).dB_t + b(X_t)dt; \quad X_0 = x.$$
A ‘diffeomorphic modification’ of \( \{X(t, x)\} \) exists ([Kunita(1997)]\(^4\)).

**Theorem**

There exists a process \( \{\tilde{X}(t, x)\}_{t \geq 0, x \in \mathbb{R}^d} \) such that

- For all \( x \in \mathbb{R}^d \), \( P(\tilde{X}(t, x, \omega) = X(t, x, \omega), t \geq 0) = 1 \).
- \( P(x \mapsto \tilde{X}(t, x, \omega) \text{ is a diffeomorphism, } \forall t \geq 0) = 1 \).

(Flow property) Let \( \theta_t : \Omega \to \Omega \) be the shift operator defined by \( (\theta_t \omega)(s) := \omega(s + t) \). Then for \( s, t \geq 0 \) we have

\[
\tilde{X}(t + s, x, \omega) = \tilde{X}(t, \tilde{X}(t, x, \omega), \theta_t \omega)
\]

for all \( x \in \mathbb{R}^d \), a.s. \( \omega \).

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- \text{(Flow property)} Let \( \theta_t : \Omega \to \Omega \) be the shift operator defined by \( (\theta_t \omega)(s) := \omega(s + t) \). Then for \( s, t \geq 0 \) we have

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\tilde{X}(t+s,x,\omega) = \tilde{X}(t,\tilde{X}(t,x,\omega),\theta_t \omega)
\]

for all \( x \in \mathbb{R}^d \), a.s. \( \omega \).

In what follows, \( \{X(t,x)\} \) will denote the modification obtained above.

Define \( X_t(\omega) : C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d) \) by \((X_t(\omega)\phi)(x) := \phi(X(t,x,\omega))\). It is a continuous linear map.

Let \( E'(\mathbb{R}^d) \) denote the space of compactly supported distributions (dual of \( C^\infty(\mathbb{R}^d) \)).

Let \( X_t(\omega)^* : E'(\mathbb{R}^d) \to E'(\mathbb{R}^d) \) be the transpose of the map \( X_t(\omega) \). Note that

\[
\langle X_t(\omega)^* \psi , \phi \rangle = \langle \psi , X_t(\omega)\phi \rangle , \forall \phi \in C^\infty(\mathbb{R}^d), \psi \in E'(\mathbb{R}^d).
\]
Define $Y_t(\omega) : \mathcal{E}'(\mathbb{R}^d) \to \mathcal{E}'(\mathbb{R}^d)$ by

$$Y_t(\omega)(\psi) := \sum_{|\alpha| \leq N} \sum_{|\gamma| \leq |\alpha|} (-1)^{|\alpha|} \int_V g_{\alpha}(x) P_\gamma \left((\partial^{\beta_1} X_1, \ldots, \partial^{\beta_d} X_d)_{|\beta| \leq |\alpha|}\right)(t, x, \omega) \partial^{\gamma} \delta X(t, x, \omega) \, dx,$$

where $P_\gamma$ are some polynomials.
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where $P_\gamma$ are some polynomials.

**Theorem**

Let $\psi \in \mathcal{E}'(\mathbb{R}^d)$. There exists $p > 0$ such that $\{Y_t(\psi)\}$ is an $S_{-p}(\mathbb{R}^d)$ valued continuous adapted process and a.s.

$$Y_t(\psi) = X_t^*(\psi), \quad t \geq 0.$$
Operators $A, L, A^*, L^*$

Now define the operators $A : C^\infty(\mathbb{R}^d) \to \mathcal{L}(\mathbb{R}^r, C^\infty(\mathbb{R}^d))$ and $L : C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$ as follows: for $\psi \in C^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

$$\begin{aligned}
A\phi &:= (A_1\phi, \cdots, A_r\phi), \\
A_i\phi(x) &:= \sum_{k=1}^d \sigma_{ki}(x) \partial_k \phi(x), \\
L\phi(x) &:= \frac{1}{2} \sum_{i,j=1}^d (\sigma^t\sigma)_{ij}(x) \partial_{ij}^2 \phi(x) + \sum_{i=1}^d b_i(x) \partial_i \phi(x).
\end{aligned}$$

We define the adjoint operators $A^* : \mathcal{E}'(\mathbb{R}^d) \to \mathcal{L}(\mathbb{R}^r, \mathcal{E}'(\mathbb{R}^d))$ and $L^* : \mathcal{E}'(\mathbb{R}^d) \to \mathcal{E}'(\mathbb{R}^d)$ as follows: for $\psi \in \mathcal{E}'(\mathbb{R}^d)$

$$\begin{aligned}
A^*\psi &:= (A_1^*\psi, \cdots, A_r^*\psi), \\
A_i^*\psi &:= -\sum_{k=1}^d \partial_k (\sigma_{ki}\psi), \\
L^*\psi &:= \frac{1}{2} \sum_{i,j=1}^d \partial_{ij}^2 ((\sigma^t\sigma)_{ij}\psi) - \sum_{i=1}^d \partial_i (b_i\psi).
\end{aligned}$$
Estimates on $A^*, L^*$ [Rajeev and Thangavelu(2008)]

Fix $p > 0$ and $q > [p] + 4$. Then there exists constants $C_1(p) > 0, C_2(p) > 0$ such that for $\psi \in \mathcal{E}'(\mathbb{R}^d) \cap S_{-p}(\mathbb{R}^d)$

$$\sum_{i=1}^{r} \|A^*_i \psi\|_{-q}^2 \leq C_1(p) \|\psi\|_{-p}^2, \quad \|L^* \psi\|_{-q} \leq C_2(p) \|\psi\|_{-p}.$$
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$$\sum_{i=1}^{r} \|A_i^* \psi\|_{-q}^2 \leq C_1(p) \|\psi\|_{-p}^2, \quad \|L^* \psi\|_{-q} \leq C_2(p) \|\psi\|_{-p}.$$

Theorem ([Rajeev and Thangavelu(2008)])

Fix $\psi \in \mathcal{E}'(\mathbb{R}^d)$. The $S_{-p}(\mathbb{R}^d)$ valued continuous adapted process $\{Y_t(\psi)\}$ satisfies the following equation in $S_{-q}(\mathbb{R}^d)$ a.s.

$$Y_t(\psi) = \psi + \int_0^t A^*(Y_s(\psi)) \cdot dB_s + \int_0^t L^*(Y_s(\psi)) \, ds, \ \forall t \geq 0.$$
Theorem ([Rajeev and Thangavelu(2008)])

Fix $p > 0$ and $q > [p] + 4$. Let $ar{\psi} \in \mathcal{E}'(\mathbb{R}^d) \cap \mathcal{S}_{-p}(\mathbb{R}^d)$. Then $\psi(t) := \mathbb{E} Y_t(\bar{\psi})$ solves

$$
\psi(t) = \bar{\psi} + \int_0^t L^* \psi(s) \, ds
$$

in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$. Moreover the solution is unique if the pair $(A^*, L^*)$ satisfies the Monotonicity inequality, viz

$$
2 \langle \phi, L^* \phi \rangle_{-q} + \sum_{i=1}^r \| A^*_i \phi \|_{-q}^2 \leq C \| \phi \|_{-q}^2, \forall \phi \in \mathcal{E}'(\mathbb{R}^d) \cap \mathcal{S}_{-p}(\mathbb{R}^d),
$$

where $C = C(p)$ is a positive constant.
Theorem ([Rajeev and Thangavelu(2008)])

Fix $p > 0$ and $q > [p] + 4$. Let $\bar{\psi} \in \mathcal{E}'(\mathbb{R}^d) \cap S_{-p}(\mathbb{R}^d)$. Then $\psi(t) := \mathbb{E} Y_t(\bar{\psi})$ solves

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where $C = C(p)$ is a positive constant.

Remark: When $\sigma = 0$, the PDE considered above reduces to linear transport equations considered in [DiPerna and Lions(1989)]\(^5\).

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• reformulated in [Gawarecki, Mandrekar, and Rajeev(2008)]\textsuperscript{7}, [Rozovskiĭ(1990)]\textsuperscript{8} for countably Hilbertian Nuclear spaces.
• Proved in [Gawarecki, Mandrekar, and Rajeev(2009)]\textsuperscript{9} when $A^*, L^*$ were constant coefficient differential operators on $S'({\mathbb{R}^d})$.

Motivation

The key observation is that if \( \{Y_t\} \) solves an SDE of the form in \( S_p(\mathbb{R}^d) \)

\[
dY_t = A(Y_s) \cdot dB_s + L(Y_s) \, ds
\]

then

\[
E \|Y_t\|_p^2 \leq \|Y_0\|_p^2 + E \int_0^t \left[ 2 \langle Y_s, L Y_s \rangle_p + \sum_{i=1}^r \|A_i(Y_s)\|_p^2 \right] ds.
\]

If above LHS of Monotonicity Inequality \( \leq C \|Y_s\|_q^2 \), then Gronwall’s Inequality alongwith \( Y_0 = 0 \) will give the uniqueness.
Let $\sigma = (\sigma_{ij})$ be a constant $d \times r$ matrix and $b = (b_1, \ldots, b_d) \in \mathbb{R}^d$.

**Theorem ([Gawarecki, Mandrekar, and Rajeev(2009)])**

For every $p \in \mathbb{R}$, $\exists$ a constant $C = C(p, d, (\sigma_{ij}), (b_j)) > 0$, such that

$$2 \langle \phi, L\phi \rangle_p + \sum_{i=1}^{r} \|A_i \phi\|^2_p \leq C.\|\phi\|^2_p, \forall \phi \in S(\mathbb{R}^d).$$

Furthermore, by density arguments the above inequality can be extended to all $\phi \in S_{p+1}(\mathbb{R}^d)$. 
Let $\sigma = (\sigma_{ij})$ be a constant $d \times r$ matrix and $b = (b_1, ..., b_d) \in \mathbb{R}^d$.

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Furthermore, by density arguments the above inequality can be extended to all $\phi \in \mathcal{S}_{p+1}(\mathbb{R}^d)$.

Remark

Monotonicity inequality holds for $(A^*, L^*)$ when $\sigma, b$ are as above.
We consider the case $r = d$.

**Theorem ([Bhar and Rajeev(2015)])**

Let $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{R}^d$ and $C = (c_{ij})$ be a real square matrix of order $d$. Let $\sigma$ be a constant function, i.e. $\sigma(x) \equiv (\sigma_{ij})$, $\forall x \in \mathbb{R}^d$ where $\sigma_{ij} \in \mathbb{R}$, $i, j = 1, \cdots, d$. Let $b = (b_1, \cdots, b_d)$ with $b(x) := \alpha + C x$, $\forall x \in \mathbb{R}^d$. Fix $p \in \mathbb{R}$. Then
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1. The maps $A_i^*$ are bounded linear operators from $S_{p+\frac{1}{2}}(\mathbb{R}^d)$ to $S_p(\mathbb{R}^d)$ and $L^*$ is a bounded linear operator from $S_{p+1}(\mathbb{R}^d)$ to $S_p(\mathbb{R}^d)$.

2. Monotonicity inequality for $A^*, L^*$ holds, i.e. there exists a positive constant $R = R(p, d, (\sigma_{ij}), (b_j))$, such that

$$2 \langle \phi, L^* \phi \rangle_p + \sum_{i=1}^{d} \|A_i^* \phi\|_p^2 \leq R \|\phi\|_p^2$$

for all $\phi \in S_{p+1}(\mathbb{R}^d)$.

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Outline

1. Schwartz Space with Hilbertian Topology
   - Hilbertian Topology and Hermite-Sobolev Spaces

2. Known results
   - Heat Equation
   - Forward Equations
   - Monotonicity inequality

3. New results
   - Ornstein-Uhlenbeck diffusion
   - Solution to SPDEs
   - Deterministic dependence on the initial condition
   - Solution to SPDEs Contd.
Ornstein-Uhlenbeck diffusion

Consider the case \( \sigma = I, \ b(x) = -x. \)

\[
dX_t = dB_t - X_t \, dt; \quad X_0 = x
\]
Ornstein-Uhlenbeck diffusion

Consider the case $\sigma = I, b(x) = -x$.

$$dX_t = dB_t - X_t \, dt; \quad X_0 = x$$

$$X(t, x) = e^{-t}x + \int_0^t e^{-(t-s)} \, dB_s, \ 0 \leq t < \infty.$$
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$$\begin{align*}
    dX_t &= dB_t - X_t \, dt; \quad X_0 = x \\
    X(t, x) &= e^{-t}x + \int_0^t e^{-(t-s)} \, dB_s, \ 0 \leq t < \infty.
\end{align*}$$

Note: $x \mapsto X(t, x, \omega)$ is an affine map and hence is a $C^\infty$ function with bounded derivatives.
Define a continuous linear map, denoted by $X_t(\omega) : S(\mathbb{R}^d) \rightarrow S(\mathbb{R}^d)$ and given by

$$(X_t(\omega)\phi)(x) := \phi(X(t,x,\omega)), x \in \mathbb{R}^d.$$ 

Let $X_t^*(\omega) : S'(\mathbb{R}^d) \rightarrow S'(\mathbb{R}^d)$ denote the transpose of the map $X_t(\omega)$. Then for any $\psi \in S'(\mathbb{R}^d)$,

$$\langle X_t^*(\psi), \phi \rangle = \langle \psi, X_t(\phi) \rangle, \forall \phi \in S(\mathbb{R}^d).$$
Fix \( \psi \in L^1(\mathbb{R}^d) \subset S'(\mathbb{R}^d) \). The identification is given by

\[
\phi \mapsto \int_{\mathbb{R}^d} \phi(x)\psi(x) \, dx.
\]

In fact \( L^1(\mathbb{R}^d) \subset S_{-p}(\mathbb{R}^d) \) for any \( p > \frac{d}{4} \).

Define \( Y_t(\omega)(\psi) := \int_{\mathbb{R}^d} \psi(x)\delta_{X(t,x,\omega)} \, dx \).
Fix $\psi \in L^1(\mathbb{R}^d) \subset S'(\mathbb{R}^d)$. The identification is given by

$$\phi \mapsto \int_{\mathbb{R}^d} \phi(x)\psi(x) \, dx.$$ 

In fact $L^1(\mathbb{R}^d) \subset S_{-p}(\mathbb{R}^d)$ for any $p > \frac{d}{4}$.

Define $Y_t(\omega)(\psi) := \int_{\mathbb{R}^d} \psi(x)\delta_{X(t,x,\omega)} \, dx$.

$Y_t(\psi)$ is a well-defined element of $S_{-p}(\mathbb{R}^d)$ for any $p > \frac{d}{4}$.

$\mathbb{E} \|Y_t(\psi)\|_{-p}^2 \leq C^2 \left( \int_{\mathbb{R}^d} |\psi(x)| \, dx \right)^2 < \infty$ for some constant $C > 0$.

Observe that

$$\langle Y_t(\psi), \phi \rangle = \int_{\mathbb{R}^d} \psi(x)\phi(X(t,x)) \, dx$$

$$= \int_{\mathbb{R}^d} \psi(x)(X_t(\phi))(x) \, dx = \langle \psi, X_t(\phi) \rangle, \forall \phi \in S(\mathbb{R}^d).$$

$Y_t(\psi) = X_t^*(\psi)$. 
Theorem ([Bhar(2016)])

Let $p > \frac{d}{4}$ and $\psi \in \mathcal{L}^1(\mathbb{R}^d)$. Then\(^a\) the $S_{-p}(\mathbb{R}^d)$ valued continuous adapted process $\{Y_t(\psi)\}$ satisfies the following equation in $S_{-p-1}(\mathbb{R}^d)$, a.s.

$$Y_t(\psi) = \psi + \int_0^t A^*(Y_s(\psi)) \, dB_s + \int_0^t L^*(Y_s(\psi)) \, ds, \ \forall t \geq 0.$$ 

This solution is also unique.

---

Theorem ([Bhar(2016)])

Let \( p > \frac{d}{4} \) and \( \psi \in L^1(\mathbb{R}^d) \). Then the \( S_{-p}(\mathbb{R}^d) \) valued continuous adapted process \( \{Y_t(\psi)\} \) satisfies the following equation in \( S_{-p-1}(\mathbb{R}^d) \), a.s.

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\]

This solution is also unique.

---


**Sketch of Proof.**

By Itô’s formula for any \( \phi \in S(\mathbb{R}^d) \), and any \( x \in \mathbb{R}^d \)

\[
(X_t(\phi))(x) = \phi(X(t, x)) = \phi(x) + \int_0^t A\phi(X(s, x)) \, dB_s + \int_0^t L\phi(X(s, x)) \, ds
\]

\[
= \phi(x) + \int_0^t (X_s(A\phi))(x) \, dB_s + \int_0^t (X_s(L\phi))(x) \, ds
\]

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Sketch of Proof (contd.)

Then for $\phi \in S(\mathbb{R}^d)$,

$$\langle Y_t(\psi) , \phi \rangle = \langle \psi , X_t(\phi) \rangle$$

$$= \left\langle \psi , \phi + \int_0^t X_s(A\phi) \cdot dB_s + \int_0^t X_s(L\phi) \, ds \right\rangle$$

$$= \langle \psi , \phi \rangle + \int_0^t \langle \psi , X_s(A\phi) \rangle \cdot dB_s + \int_0^t \langle \psi , X_s(L\phi) \rangle \, ds$$

$$= \langle \psi , \phi \rangle + \int_0^t \langle A^* Y_s(\psi) , \phi \rangle \cdot dB_s + \int_0^t \langle L^* Y_s(\psi) , \phi \rangle \, ds$$

$$= \left\langle \psi + \int_0^t A^* Y_s(\psi) \cdot dB_s + \int_0^t L^* Y_s(\psi) \, ds , \phi \right\rangle$$
Sketch of Proof (contd.)

Then for $\phi \in \mathcal{S}(\mathbb{R}^d)$,

$$\langle Y_t(\psi), \phi \rangle = \langle \psi, X_t(\phi) \rangle$$

$$= \left\langle \psi, \phi + \int_0^t X_s(A\phi). dB_s + \int_0^t X_s(L\phi) \, ds \right\rangle$$

$$= \langle \psi, \phi \rangle + \int_0^t \langle \psi, X_s(A\phi) \rangle . dB_s + \int_0^t \langle \psi, X_s(L\phi) \rangle \, ds$$

$$= \langle \psi, \phi \rangle + \int_0^t \langle A^* Y_s(\psi), \phi \rangle . dB_s + \int_0^t \langle L^* Y_s(\psi), \phi \rangle \, ds$$

$$= \left\langle \psi + \int_0^t A^* Y_s(\psi). dB_s + \int_0^t L^* Y_s(\psi) \, ds , \phi \right\rangle$$

Proof of uniqueness: Gronwall’s inequality + Monotonicity inequality
Theorem (B.)

Let \( p > \frac{d}{4} \) and \( \psi \in L^1(\mathbb{R}^d) \). Then \( \bar{\psi}(t) := \mathbb{E} Y_t(\psi) \) solves the equation

\[
\frac{d}{dt} \bar{\psi} = L^* \bar{\psi},
\]

i.e. the equality

\[
\mathbb{E} Y_t(\psi) = \psi + \int_0^t L^* (\mathbb{E} Y_s(\psi)) \, ds
\]

holds in \( S_{-p-1}(\mathbb{R}^d) \). Furthermore this is the unique solution.
Consider random fields which arise as solutions of SDEs:

\[
dX_t = \sigma(X_t) \, dB_t + b(X_t) \, dt, \quad \forall t \geq 0; \quad X_0 = X
\]

where the coefficients \( \sigma = (\sigma_{ij}) \), \( b = (b_i) \), \( 1 \leq i, j \leq d \) are Lipschitz continuous and the random variable \( X \) is independent of the Brownian motion \( \{B_t\} \). For any \( x \in \mathbb{R}^d \), let \( \{X_t^x\} \) denote the solution of the SDE with \( X_0 = x \).
Consider random fields which arise as solutions of SDEs:

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It is known that the solutions to such equations are Gaussian if \( X \) is Gaussian (or a constant), \( \sigma \) is a constant \( d \times d \) matrix and \( b(x) = a + bx, \forall x \in \mathbb{R}^d \) for some \( a \in \mathbb{R}^d, b \in \mathbb{R} \).
Consider random fields which arise as solutions of SDEs:

\[ dX_t = \sigma(X_t) \, dB_t + b(X_t) \, dt, \quad \forall t \geq 0; \quad X_0 = X \]

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The strong solutions of the above equations are maps \( F : [0, \infty) \times \mathbb{R}^d \times C([0, \infty), \mathbb{R}^d) \rightarrow \mathbb{R}^d \) such that the solutions with initial value \( X \) and Brownian motion \( \{B_t\} \) is given at time \( t \) by

\[ X_t = F(t, X, B), \text{ a.s.} \]
In the Gaussian case as above (i.e. $\sigma$ is constant and $b(x) = a + bx$) it is known that a.s.

$$F(t, x, B) = e^{tb}x + (e^{tb} - 1)b^{-1}a + \int_0^t e^{(t-s)b} \sigma dB_s.$$ 

We wish to characterize the maps $F$ for which the solutions of the above SDEs are Gaussian.
We make a definition of the class of SDEs such that \( \{X_t^x\} \) has a deterministic ‘local’ component.

**Definition**

*We say the general solution of the SDE depends deterministically on the initial condition, if there exists a function \( f : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) such that for any \( x \in \mathbb{R}^d \), we have a.s.*

\[
X_t^x(\omega) = f(t, x) + X_t^0(\omega), \quad t \geq 0.
\]

**Remark:** In this case, for every fixed \( x \in \mathbb{R}^d \), the map
\[
t \mapsto \frac{\partial f}{\partial t}(t, x) = (\frac{\partial f_1}{\partial t}(t, x), \ldots, \frac{\partial f_d}{\partial t}(t, x))
\]

is continuous.
Theorem ([Bhar(2016)])

Let $\sigma, b$ be Lipschitz continuous functions. Suppose the following happen$^a$:

1. There exists an $x \in \mathbb{R}^d$ such that the determinant of $(\sigma_{ij}(x))$ is not zero,
2. $b_i \in C^1(\mathbb{R}^d, \mathbb{R})$, $i = 1, \cdots, d$ where $b = (b_1, \cdots, b_d)$,
3. For every fixed $x \in \mathbb{R}^d$, the map $t \in [0, \infty) \mapsto \frac{\partial f}{\partial t}(t, x)$ is of bounded variation.

Then the general solution of the SDE depends deterministically on the initial condition if and only if $\sigma$ is a real non-singular matrix of order $d$ and $b$ is of the form $b(x) = \alpha + Cx$ and $f(t, x) = e^{tC}x$ where $\alpha \in \mathbb{R}^d$ and $C$ is a real square matrix of order $d$.

Proposition (B.)

Let $\sigma, b$ be Lipschitz continuous functions.

1. **Suppose** the general solution of the SDE depends deterministically on the initial condition, where the function $f$ has the decomposition $f(t, x) = g(t)h(x)$ with $g \in C^1([0, \infty), \mathbb{R})$, $h : \mathbb{R}^d \to \mathbb{R}^d$. Then $f(t, x) = \tilde{g}(t)x$ for some $\tilde{g} \in C^1([0, \infty), \mathbb{R})$ with $\tilde{g}(0) = 1$.

2. The solution to the SDE depends deterministically on the initial condition in the following form: for each $x \in \mathbb{R}^d$, a.s. $t \geq 0$

$$X_t^x = g(t)x + X_t^0,$$

for some $g \in C^1([0, \infty), \mathbb{R})$ with $g(0) = 1$ if and only if $\sigma$ is a constant $d \times d$ matrix, $b(x) = \alpha + \beta x$ and $g(t) = e^{\beta t}$, $t \geq 0$ where $\alpha \in \mathbb{R}^d$, $\beta \in \mathbb{R}$. In this case, the solution has the form

$$X_t^x = \begin{cases} e^{\beta t}x + \sigma \int_0^t e^{\beta(t-s)} dB_s + \frac{e^{\beta t} - 1}{\beta} \alpha, & \text{if } \beta \neq 0 \\ x + t\alpha + \sigma B_t, & \text{if } \beta = 0. \end{cases}$$
Let $\sigma = (\sigma_{ij})$ be a real square matrix of order $d$.

Let $b = (b_1, \cdots, b_d)$ be of the form $b(x) = \alpha + Cx$ where $\alpha \in \mathbb{R}^d$ and $C$ is a real square matrix of order $d$. 

Let $\sigma = (\sigma_{ij})$ be a real square matrix of order $d$.

Let $b = (b_1, \cdots, b_d)$ be of the form $b(x) = \alpha + Cx$ where $\alpha \in \mathbb{R}^d$ and $C$ is a real square matrix of order $d$.

Define the continuous linear maps $X_t(\omega) : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ and $X_t^*(\omega) : S'(\mathbb{R}^d) \to S'(\mathbb{R}^d)$.

For $\psi \in \mathcal{L}^1(\mathbb{R}^d)$, define $Y_t(\psi)$ as before. Then $Y_t(\psi) = X_t^*(\psi)$. 
Let $\sigma = (\sigma_{ij})$ be a real square matrix of order $d$. Let $b = (b_1, \cdots, b_d)$ be of the form $b(x) = \alpha + Cx$ where $\alpha \in \mathbb{R}^d$ and $C$ is a real square matrix of order $d$.

Define the continuous linear maps $X_t(\omega) : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ and $X_t^*(\omega) : S'(\mathbb{R}^d) \to S'(\mathbb{R}^d)$.

For $\psi \in L^1(\mathbb{R}^d)$, define $Y_t(\psi)$ as before. Then $Y_t(\psi) = X_t^*(\psi)$.

**Theorem (B.)**

Let $p > \frac{d}{4}$ and $\psi \in L^1(\mathbb{R}^d)$. Then the $S_{-p}(\mathbb{R}^d)$ valued continuous adapted process $\{Y_t(\psi)\}$ satisfies the following equation in $S_{-p-1}(\mathbb{R}^d)$, a.s.

$$Y_t(\psi) = \psi + \int_0^t A^*(Y_s(\psi)) \, dB_s + \int_0^t L^*(Y_s(\psi)) \, ds, \quad \forall t \geq 0.$$ 

This is also the unique solution of the previous equation. Furthermore,

$$\mathbb{E} Y_t(\psi) = \psi + \int_0^t L^* \mathbb{E} Y_s(\psi) \, ds$$

holds in $S_{-p-1}(\mathbb{R}^d)$. Furthermore this is the unique solution.
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